An index theorem for operators with horn singularities

DISSERTATION

zur Erlangung des akademischen Grades Dr. rer. nat.
im Fach Mathematik

eingereicht an der
Mathematisch-Naturwissenschaftlichen Fakultät II
der Humboldt-Universität zu Berlin

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eingereicht am: 31. 1. 2013
Tag der Verteidigung: 25. 10. 2013

Abstract

The closed extensions of geometric operators (Spin-Dirac, Gauss-Bonnet and Signature operator) on a manifold with metric horns are Fredholm operators, and their indices were computed by Matthias Lesch, Norbert Peyerimhoff and Jochen Brüning. It was shown that the restrictions of all three operators to a punctured neighbourhood of the singular point are unitary equivalent to a class of irregular singular operator-valued differential operators of first order. The solution operators of the corresponding differential equations defined a parametrix which was applied to prove the Fredholm property.

In this thesis a class of irregular singular differential operators of first order - called horn operators - is introduced that extends the examples mentioned above. It is proved that an elliptic differential operator of first order whose restriction to the neighbourhood of the singular point is unitary equivalent to a horn operator is Fredholm and its index is computed. Finally, this abstract index theorem is applied to compute the indices of geometric operators on manifolds with multiply warped product singularities that extend the notion of metric horns considerably.

Zusammenfassung

Die abgeschlossenen Erweiterungen der sogenannten geometrischen Operatoren (Spin-Dirac, Gauß-Bonnet und Signatur-Operator) auf Mannigfaltigkeiten mit metrischen Hörnern sind Fredholm-Operatoren und ihr Index wurde von Matthias Lesch, Norbert Peyerimhoff und Jochen Brüning berechnet. Es wurde gezeigt, dass die Einschränkungen dieser drei Operatoren auf eine punktierte Umgebung des singulären Punkts unitär äquivalent zu irregulär singulären Operator-wertigen Differentialoperatoren erster Ordnung sind. Die Lösungsoperatoren der dazugehörigen Differentialgleichungen definierten eine Parametrix, mit deren Hilfe die Fredholmeigenschaft bewiesen wurde.

In der vorliegenden Doktorarbeit wird eine Klasse von irregulären singulären Differentialoperatoren erster Ordnung, genannt Horn-Operatoren, eingeführt, die die obigen Beispiele verallgemeinern. Es wird bewiesen, dass ein elliptischer Differentialoperator erster Ordnung, dessen Einschränkung auf eine punktierte Umgebung des singulären Punkts unitär äquivalent zu einem Horn-Operator ist, Fredholm ist, und sein Index wird berechnet. Schließlich wird dieser abstrakte Index-Satz auf geometrische Operatoren auf Mannigfaltigkeiten mit „multiply warped product"-Singularitäten angewendet, welche eine wesentliche Verallgemeinerung der metrischen Hörner darstellen.

Contents

1 Introduction

We consider an open smooth manifold $M = U \cup M_1$, where $U = (0, s_0] \times N$ is a punctured neighbourhood of the singularity and M_1 a compact manifold with boundary N smoothly attached at $\{s_0\} \times N$. Since the singularity at $\{0\} \times N$ is left out, M is a non-complete manifold. Furthermore, we assume that there are vector bundles E and F over M and an elliptic differential operator of first order

$$D : \Gamma(E) \to \Gamma(F).$$

The aim of this work is to describe singular settings in which D is a Fredholm operator and to compute its index.

We begin by discussing the Gauss-Bonnet operator over a manifold with a metric horn singularity. In [Che80], Jeff Cheeger introduced metric horns

$$U = ((0, s_0] \times N, dr^2 \oplus r^{2\beta} g_N), \quad \beta > 1,$$

where (N, g_N) is a closed Riemannian manifold. In [BS88] and [LP98], it is shown that the Gauss-Bonnet operator

$$D^{GB} = d + \delta : \Omega^{ev}(M) \to \Omega^{odd}(M)$$

is unitary equivalent to

$$D^{GB}\big|_U \cong \frac{\partial}{\partial x} + \frac{1}{x^\beta} D_N + \frac{\beta}{x} A : \Omega(N) \to \Omega(N),$$

where $D_N = d_N + \delta_N$ and $A\alpha_j = (-1)^j (j - \frac{n}{2})$ for $\alpha_j \in \Omega^j(N)$. Let $\mathcal{H}(N)$ be the finite-dimensional space of harmonic forms on the closed manifold N. Then the Gauss-Bonnet operator decomposes into

$$D^{GB}\big|_U \cong \left(\frac{\partial}{\partial x} + \frac{1}{x^\beta} D_N + \frac{\beta}{x} A\right)\bigg|_{\mathcal{H}(N)^\perp} \oplus \left(\frac{\partial}{\partial x} + \frac{\beta}{x} A\right)\bigg|_{\mathcal{H}(N)}. \tag{1.1}$$

The restriction to $\mathcal{H}(N)^\perp$ is an operator-valued differential operator with horn singularity. The restriction to $\mathcal{H}(N)$ is a matrix-valued differential operator with cone singularity. It is worth pointing out that the eigenvalues of βA between $-\frac{1}{2}$ and $\frac{1}{2}$ determine the number of closed extensions and enter the index formula.

To prove Fredholm properties of the Spin-Dirac, Gauss-Bonnet and Signature operator on manifolds with metric horn singularities, Matthias Lesch and Norbert Peyerimhoff introduced abstract operators generalizing (1.1) in [LP98]. The aim of this thesis is to extend their definition by allowing similar variation as in [Brü92] on the operator-valued part. This approach leads us to the definition of a horn operator

$$\left(\frac{\partial}{\partial x} + \frac{1}{x^\beta} S(x) + S_1(x)\right) \oplus_{H \oplus \widetilde{H}} \left(\frac{\partial}{\partial x} + \frac{1}{x}\widetilde{S} + \widetilde{S}_1(x)\right),$$

where H and $\widetilde{H}$ are Hilbert spaces and $\dim \widetilde{H} < \infty$. Furthermore, we assume the following properties:

1. The family $x \mapsto S(x)$ of self-adjoint operators is strongly differentiable on a common dense domain $H_1 \subset H$, $x \mapsto S(x)S(s_0)^{-1}$ is continuous in norm,
$$|S(x)| \geqslant C_2 > 0 \quad \text{and} \quad \int_0^{s_0} x^\beta \left\|S'(x)S(x)^{-1}\right\|^2_{\mathcal{L}(H)} dx < \infty.$$
2. $\widetilde{S}$ is a symmetric matrix on $\widetilde{H}$.
3. The families $I \ni x \mapsto S_1(x) \in \mathcal{L}(H)$ and $I \ni x \mapsto \widetilde{S}_1(x) \in \mathcal{L}(\widetilde{H})$ satisfy
$$\int_0^{s_0} x^\beta \left\|S_1(x)\right\|^2_{\mathcal{L}(H)} + x|\log x| \left\|\widetilde{S}_1(x)\right\|^2_{\mathcal{L}(\widetilde{H})} dx < \infty.$$

Let us proceed by comparing the properties of the abstract operators defined in [Brü92], Lemma 2.2, and in [LP98], page 659 and the horn operator defined above. These three operators have the common normal form

$$D|_U \cong \left(\frac{\partial}{\partial x} + x^{-\beta} S(x) + S_1(x)\right) \oplus \left(\frac{\partial}{\partial x} + x^{-1}\widetilde{S}(x) + \widetilde{S}_1(x)\right).$$

The respective assumptions on β, S, S_1, $\widetilde{S}$ and $\widetilde{S}_1$ are compared in the following table.

	[Brü92]	[LP98]	Thesis
β	$\beta = 1$	$\beta > 1$	$\beta > 1$
S	$\lvert S(x)\rvert \geqslant C_2 > \frac{1}{2}$ $\int_0^{s_0} x \left\|S'(x)S(x)^{-1}\right\|^2 dx < \infty$	$S(x) \equiv S(s_0)$ for x small.	$\lvert S(x)\rvert \geqslant C_2 > 0$ $\int_0^{s_0} x^\beta \left\|S'(x)S(x)^{-1}\right\|^2 dx < \infty$
S_1	0	$\left\|S_1(x)\right\| \leqslant C$	$\int_0^{s_0} x^\beta \left\|S_1(x)\right\|^2 dx < \infty$
$\widetilde{S}$	$\lvert\widetilde{S}(x)\rvert \leqslant C_1 < \frac{1}{2}$ $\int_0^{s_0} x \left\|\widetilde{S}'(x)\widetilde{S}(x)^{-1}\right\|^2 dx < \infty$	$\widetilde{S}(x) \equiv \widetilde{S}(s_0)$ for x small.	$\widetilde{S}(x) \equiv \widetilde{S}(s_0)$ for x small.
$\widetilde{S}_1$	0	0	$\int_0^{s_0} x\lvert\log x\rvert \left\|\widetilde{S}_1(x)\right\|^2 dx < \infty$

A first order differential operator on a manifold $M = U \cup M_1$ where $D|_U$ is unitary equivalent to a horn operator is called an operator with horn singularity. For such an operator our main theorem states that $D_{\min}$, $D_{\max}$ and the closed extensions in between defined by

$$W \subset \mathcal{D}(D_{\max}) \Big/ \mathcal{D}(D_{\min}) \cong \bigoplus_{|\lambda|<\frac{1}{2}} \ker(\widetilde{S} - \lambda),$$

$$\mathcal{D}(D_W) := \mathcal{D}(D_{\min}) \oplus W, \quad D_W := D_{\max}|_{\mathcal{D}(D_W)}$$

are Fredholm operators and their indices are given by

$$\begin{aligned} \operatorname{ind} D_W = \int_{M_1} \omega_D - \frac{1}{2}\Bigg(\eta(S(s_0)) + \sum_{\lambda \geqslant 0} \dim\ker(\widetilde{S} - \lambda) - \sum_{\lambda<0} \dim\ker(\widetilde{S} - \lambda) \Bigg) \\ - \sum_{-\frac{1}{2}<\lambda<0} \dim\ker(\widetilde{S} - \lambda) + \sum_{k=1}^{\dim N} \widetilde{\alpha}_k \operatorname{Res}_k \eta_{S(s_0)} + \dim W, \end{aligned}$$

where $\eta_{S(s_0)}$ is the eta function, $\eta(S(s_0))$ is the eta-invariant of $S(s_0)$, ω_D denotes the index form of D and $\widetilde{\alpha}_k \in \mathbb{R}$. It shall not be concealed that, unlike in the cone case treated in [BS88], no explicit formulas for the constants $\widetilde{\alpha}_k$ are given. Therefore, the index theorem only makes sense if $\operatorname{Res}_k \eta_{S(s_0)} = 0$ for all $k \in \mathbb{N}$. Fortunately, a large class of operators on Riemannian manifolds has this property (confer [ABP73] and [ABP75]).

The outline of the thesis is as follows. The second chapter introduces the notion of horn operators, of operators with horn singularities, and states the index theorem. For every x the operator $S(x)$ is self-adjoint by assumption. By the spectral theorem the projections $Q_>(x)$ and $Q_<(x)$ corresponding to the intervals $[C_2, \infty)$ and $(-\infty, -C_2]$, respectively, exist for every x. In Section 2.2 it is proved that, under the assumption that $x \mapsto S'(x)S(s_0)^{-1}$ is continuous in norm, the families $x \mapsto Q_>(x)$ and $x \mapsto Q_<(x)$ are differentiable in norm and their derivatives inherit norm estimates from S'.

Chapter 3 contains the proof of the index theorem for operators D with horn singularity

$$D|_U \cong \left(\frac{\partial}{\partial x} + x^{-\beta} S(x) + S_1(x) \right) \oplus \left(\frac{\partial}{\partial x} + x^{-1} \widetilde{S} + \widetilde{S}_1(x) \right).$$

In the first two sections it is shown that the operators in question are Fredholm, and in the last two sections their indices are computed.

In Section 3.1 a parametrix for the "reduced" horn operator

$$\left(\frac{\partial}{\partial x} + x^{-\beta} S(x) \right) \oplus \left(\frac{\partial}{\partial x} + x^{-1} \widetilde{S} \right), \quad Q_>(x) \equiv Q_>(s_0) \text{ and } Q_<(x) \equiv Q_<(s_0)$$

is constructed. The existence of the parametrix follows from the theory of operator-valued first order differential operators as treated in [Kre71] (see also Section 4.2). The rest of the section is devoted to the computations of estimates and properties of this parametrix.

In the second section it is shown that any horn operator can be transformed into a "reduced" horn operator as defined in the first section. The transformation function, the parametrix of the "reduced" operator and an interior parametrix on M_1 are combined to prove that $D_{\min}$, $D_{\max}$ and all closed extensions in between are Fredholm.

In the last two sections it is proved that the index stays the same as in the cone case. The first step is accomplished in Section 3.3, where an index preserving homotopy is defined from the "reduced" horn operator to the "constant" horn operator

$$\left(\frac{\partial}{\partial x} + x^{-\beta} S(s_0)\right) \oplus \left(\frac{\partial}{\partial x} + x^{-1} \widetilde{S}\right).$$

The last step of the proof – presented in Section 3.4 – correlates the index of the horn operator with the index of the cone operator

$$\frac{\partial}{\partial x} + x^{-1} S_0 \oplus \widetilde{S}.$$

In Chapter 4 we review three methods that have been applied in the proof of the index theorem. In the first section an index theorem for families of densely defined operators with variable domain is proved. This result goes back to [CL63], [Brü92] and [LP98].

In Section 4.2 the solution operators of an evolution equation

$$\frac{\partial}{\partial t} + A(t) = 0,$$

where $t \to A(t)$ is a strongly continuous family of densely defined operators with common domain, is constructed. This can be found in [Kre71], Chapter 3.

In the last two sections the asymptotic expansion of the heat kernel for second order differential operators is computed. This exposition follows [Brü88], Chapter 2 and 4.

In the last chapter the theory of horn operators is applied to prove index theorems for the Spin-Dirac, Gauss-Bonnet and Signature operator on a manifold with a multiply warped product singularity

$$U = \Big((0, s_0] \times N_1 \times \cdots \times N_l, dr^2 \oplus h_1^2(r) g_1 \oplus \cdots \oplus h_l^2(r) g_l\Big),$$

where the warping functions $h_q \in C^1((0, s_0], (0, \infty))$ and (N_q, g_q) are closed Riemannian manifolds.

I wish to express my gratitude to my supervisor Jochen Brüning, for introducing me to the fascinating subject of index theory on singular manifolds. In a long row of lectures, seminars and personal discussions he taught me a major part of my mathematical knowledge. I gratefully acknowledge the financial support of the international research training group "Geometry and Analysis" (GRK 870) and the collaborative research centre "Space · Time · Matter" (SFB 647). I am deeply indebted to Martin Altmann for his tireless effort and endless patience. Furthermore, I would like to thank Klaus Mohnke, Dorothee Schüth, Helga Baum, Jörn Müller, Alexandre Krestiachine, Sebastian Boldt and Batu Güneysu for many helpful and motivating conversations. I would like to express my gratitude to Kati Blaudzun for help and assistance. Finally, I want to thank Sylvia Lapp and Jean-Marleen Majewski for their support and affection.

List of notations

C	The actual value of C may change from line to line. So $f_1(x) \leqslant Cf_2(x)$ has to be read in the way $f_1(x) = O(f_2(x))$. This abuse of notation is necessary to keep the sheer amount of indices at bay.
C_2, C_4	Universal constants (see Definition 2.1.1 on page 8)
horn operator	A singular differential operator (see Definition 2.1.1 on page 8)
HO^2, HO^∞	Subclasses of the horn operator that satisfy an L^2 and L^∞-estimate, respectively (see Definition 2.1.1 on page 8)
operator with (C^1-)horn singularity	An elliptic differential operator of first order on $M = M_1 \cup U$, where $D\vert_U$ is unitary equivalent to a horn operator (see Definition 2.1.2 on page 9)
h_q, $q = 1, \ldots, l$	Warping functions of the multiply warped product (see Definition 5.1.1 on page 95)
$\mathcal{L}(H_1, H_2)$	Set of bounded operators between the Hilbert spaces H_1 and H_2
$\mathcal{L}(H)$	Set of bounded operators from H to H
$M = M_1 \cup U$, $U = I \times N$	M, M_1, N are smooth manifolds, M is open and M_1 is compact with closed boundary N. $I = (0, s_0]$ for an $s_0 \in (0, \infty)$.
$\Omega(M)$	The vector bundle of differential forms on a manifold M
$Q_>,\ Q_<,\ Q_{\pm\frac{1}{2}}$	For every $x \in I$, $Q_>(x)$ and $Q_<(x)$ are the spectral projections of $S(x)$ in H corresponding to $[C_2, \infty)$ and $(-\infty, -C_2]$, respectively. $Q_{+\frac{1}{2}}$ and $Q_{-\frac{1}{2}}$ are the projections from $\widetilde{H}$ to $\ker(\widetilde{S} - \frac{1}{2})$ and $\ker(\widetilde{S} - \frac{1}{2})$, respectively.
$\Gamma(E)$	The sections of a vector bundle E

2 Statement of the index theorem

2.1 Definition and theorem

Definition 2.1.1. Let $H_{\mathrm{tot}} = H \oplus \widetilde{H}$ be a Hilbert space, where $\widetilde{H}$ is finite-dimensional. Let $H_1 \subset H$ be a dense subspace and $I = (0, s_0]$. A *horn operator* $\mathrm{T}_{\mathrm{tot}} = T \oplus \widetilde{T} : C^1(I, H_1 \oplus \widetilde{H}) \to L^2(I, H \oplus \widetilde{H})$ decomposes into

$$\begin{aligned} T &= \frac{\partial}{\partial x} + x^{-\beta} S(x) + S_1(x), \quad \beta > 1, \\ \widetilde{T} &= \frac{\partial}{\partial x} + x^{-1} \widetilde{S} + \widetilde{S}_1(x) \end{aligned}$$

with the following properties:

1. *Regularity:* $I \ni x \mapsto S(x) \in \mathcal{L}(H_1, H)$ is a strongly continuously differentiable family of self-adjoint operators with common domain H_1.

2. *Spectral gap:* There is a constant $C_2 > 0$, such that $|S(x)| \geqslant C_2$.

3. *Spectral projections:* For $x \in I$ let $Q_>(x)$ and $Q_<(x)$ denote the spectral projections of $S(x)$ corresponding to the intervals $[C_2, \infty)$ and $(-\infty, C_2]$, respectively. We assume that they are continuously differentiable in norm and satisfy the inequality

$$\int_0^{s_0} x^\beta \left(\left\| Q'_>(x) \right\|^2_{\mathcal{L}(H)} + \left\| Q'_<(x) \right\|^2_{\mathcal{L}(H)} \right) dx < \infty.$$

4. *Variation:* The function $\alpha(x) := \left\| S'(x) S(x)^{-1} \right\|_{\mathcal{L}(H)}$ satisfies either

$$\begin{aligned} C_4^2(s_0) &:= \int_0^{s_0} x^\beta \alpha^2(x) dx < \infty && (\mathrm{HO}^2 \text{ case}) && \text{or} \\ C_4(s) &:= \sup_{x \in (0,s]} x^\beta \alpha(x) \overset{s \to 0}{\longrightarrow} 0 && (\mathrm{HO}^\infty \text{ case}). \end{aligned}$$

5. *Perturbation S_1:* The family $I \ni x \mapsto S_1(x) \in \mathcal{L}(H)$ satisfies

$$\int_0^{s_0} x^\beta \left\| S_1(x) \right\|^2_{\mathcal{L}(H)} dx < \infty.$$

6. *Cone part $\widetilde{S}$:* $\widetilde{S}$ is a symmetric matrix on $\widetilde{H}$.

7. *Perturbation* $\widetilde{S}_1$*:* Let $Q_{\frac{1}{2}}$ and $Q_{-\frac{1}{2}}$ be projections to the eigenspaces of $\widetilde{S}$ corresponding to the eigenvalues $\frac{1}{2}$ and $-\frac{1}{2}$, respectively. The family $x \mapsto \widetilde{S}_1(x) \in \mathcal{L}(\widetilde{H})$ satisfies

$$\int_0^{s_0} x \left\| \widetilde{S}_1(x) \left(\mathbb{1} - Q_{\frac{1}{2}} - Q_{-\frac{1}{2}} \right) \right\|^2_{\mathcal{L}(\widetilde{H})} + x|\log x| \left\| \widetilde{S}_1(x) \left(Q_{\frac{1}{2}} + Q_{-\frac{1}{2}} \right) \right\|^2_{\mathcal{L}(\widetilde{H})} dx < \infty.$$

Definition 2.1.2. Let M be a smooth manifold with an open subset U, such that $M_1 := M \backslash U$ is a compact manifold with boundary N. Let E and F be vector bundles over M and $D : H^1(M, E) \to L^2(M, F)$ a first order elliptic differential operator. Assume that there are unitary maps

$$\begin{aligned} \Phi :\quad & L^2(U, F|_U) \to L^2(I, H_{\mathrm{tot}}) \qquad \text{and} \\ \Phi_1 :\quad & H^1(U, E|_U) \to H^1(I, H_{\mathrm{tot}}) \cap L^2(I, H_1 \oplus \widetilde{H}), \end{aligned}$$

such that $\Phi D|_U \Phi_1^*$ is a horn operator. Such an operator D is called an *operator with horn singularity.*

An operator with horn singularity where S is a family of elliptic differential operators of first order over a suitable bundle on N and $x \mapsto S(x)S(s_0)^{-1}$ is continuous in norm is called an *operator with* C^1*-horn singularity.*

Theorem 2.1.3. *Let D be an operator with C^1-horn singularity. There are constants $\widetilde{\alpha}_k \in \mathbb{R}$, such that*

$$\begin{aligned} \operatorname{ind} D_{\min} = \int_{M_1} \omega_D - \frac{1}{2} \left(\eta(S(s_0)) + \sum_{\lambda \geqslant 0} \dim\ker\left(\widetilde{S} - \lambda\right) - \sum_{\lambda < 0} \dim\ker\left(\widetilde{S} - \lambda\right) \right) \\ - \sum_{-\frac{1}{2} < \lambda < 0} \dim\ker\left(\widetilde{S} - \lambda\right) + \sum_{k=1}^{\dim N} \widetilde{\alpha}_k \operatorname{Res}_k \eta_{S(s_0)}, \end{aligned}$$

where $\eta_{S(s_0)}$ is the eta-function and $\eta(S(s_0))$ the eta-invariant of $S(s_0)$, and ω_D denotes the index form of D, i.e. $\omega_D(p)$ is the constant term in the asymptotic expansion of

$$\operatorname{tr}\left(\varphi e^{-tD^*D}(p,p) - \varphi e^{-tDD^*}(p,p)\right), \quad p \in M \text{ as } t \to 0,$$

and the integral stands for a certain regularization of the possibly divergent integral. The closed extensions of $D_{\min}$ are defined by

$$W \subset \mathcal{D}(D_{\max}) \big/ \mathcal{D}(D_{\min}) \cong \bigoplus_{|\lambda| < \frac{1}{2}} \ker(\widetilde{S} - \lambda),$$

$$\mathcal{D}(D_W) := \mathcal{D}(D_{\min}) \oplus W, \quad D_W := D_{\max}|_{\mathcal{D}(D_W)}.$$

The extensions are all Fredholm and their indices are $\operatorname{ind} D_W = \operatorname{ind} D_{\min} + \dim W$.

Proof. This theorem summarizes Theorem 3.2.5, Theorem 3.4.3 and Corollary 3.4.4. $\square$

Remarks:

1. If D is only an operator with horn singularity, then Theorem 3.2.5 shows that $D_{\min}$ and $D_{\max}$ and all the closed extensions in between are Fredholm operators.
2. Unlike in the article [BS88], where the coefficients in front of the eta residua are explicitly computable, the above theorem only makes sense if $\operatorname{Res}_k \eta_{S(s_0)} = 0$ for $k \in \mathbb{N}$. Fortunately, a large class of operators on Riemannian manifolds has this property (confer [ABP73] and [ABP75]).

2.2 Regularity of spectral projections

In general, it is complicated to compute the spectral projections $Q_>(x)$ and $Q_<(x)$. Fortunately, if $x \mapsto S'(x)S(s_0)^{-1}$ is continuous in norm and S satisfies the HO^2 estimate, it is possible to eliminate the assumptions on the spectral projections (Definition 2.1.1, 3.) completely.

Lemma 2.2.1. [1] *Consider a family of self-adjoint operators $[a,b] \ni x \mapsto A(x)$ in H with common dense domain H_1 and $z_0 \in \mathbb{R} \setminus \operatorname{spec} A(x)$ for all $x \in [a,b]$. If A is strongly differentiable on $[a,b]$ and if for an $x_0 \in [a,b]$ the family $x \mapsto A'(x)(A(x_0) - z_0)^{-1}$ is continuous in norm, then the integral function*

$$[a,b] \ni x \mapsto I(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} (A(x) - z_0 - it)^{-1}\, dt$$

is continuously differentiable in norm with derivative

$$I'(x) = J(x) := -\frac{1}{\pi} \int_{-\infty}^{\infty} (A(x) - z_0 - it)^{-1} A'(x) \, (A(x) - z_0 - it)^{-1}\, dt$$

and satisfies $\|I'(x)\|_{\mathcal{L}(H)} \leqslant \left\| A'(x) \, (A(x) - z_0)^{-1} \right\|_{\mathcal{L}(H)}$.

Proof. We assume without loss of generality that $z_0 = 0$. We begin with a number of norm estimates:

1. $x \mapsto A'(x)A(x_0)^{-1}$ *is uniformly bounded:* $[a,b] \ni x \mapsto A'(x)A(x_0)^{-1} \in \mathcal{L}(H)$ is strongly continuous on a compact interval. Thus, the theorem of Banach-Steinhaus yields
$$\left\|A'(x)A(x_0)^{-1}\right\|_{\mathcal{L}(H)} \leqslant M_1 \quad \text{for all } x \in [a,b].$$
2. $x \mapsto A(x_0)A(x)^{-1}$ *is uniformly bounded:*

$$\begin{aligned}
& && A \text{ is strongly continuously differentiable.} \\
&\Rightarrow && x \mapsto A(x)A(x_0)^{-1} \in \mathcal{L}(H) \text{ is strongly continuously differentiable.} \\
&\Rightarrow && x \mapsto A(x)A(x_0)^{-1} \text{ is continuous in norm (Lemma 3.5 in [Kre71]).} \\
&\Rightarrow && x \mapsto A(x_0)A(x)^{-1} = \left(A(x)A(x_0)^{-1}\right)^{-1} \text{ is continuous in norm.} \\
&\Rightarrow && \left\|A(x_0)A(x)^{-1}\right\|_{\mathcal{L}(H)} \leqslant M_2 \quad \text{for all } x \in [a,b].
\end{aligned}$$

[1] This lemma and its proof is inspired by Lemma A.1 on page 622 in [BB01].

3. $\left\| |A(x)|^{-\frac{1}{2}} A'(x) |A(x)|^{-\frac{1}{2}} \right\|_{\mathcal{L}(H)} \leqslant \left\| A'(x)A(x)^{-1} \right\|_{\mathcal{L}(H)}$ *and is uniformly bounded:*
   Since A is strongly continuously differentiable, Lemma 1.9 on page 186 in [Kre71] shows
$$\begin{aligned}\left\| |A(x)|^{-\frac{1}{2}} A'(x) |A(x)|^{-\frac{1}{2}} \right\|_{\mathcal{L}(H)} &\leqslant \left\| A'(x)A(x)^{-1} \right\|_{\mathcal{L}(H)} \\ &\leqslant \left\| A'(x)A(x_0)^{-1} \right\|_{\mathcal{L}(H)} \left\| A(x_0)A(x)^{-1} \right\|_{\mathcal{L}(H)} \leqslant M_1 M_2 =: M_3.\end{aligned}$$

4. $\left\| |A(w)|^{-\frac{1}{2}} \left[A'(x) - A'(y)\right] |A(z)|^{-\frac{1}{2}} \right\|_{\mathcal{L}(H)} \leqslant M_2 \left\| \left[A'(x) - A'(y)\right] A(x_0)^{-1} \right\|_{\mathcal{L}(H)}$:
   Since A is strongly continuously differentiable, the proof of Lemma 1.9 on page 186 in [Kre71] shows
$$\begin{aligned}&\left\| |A(w)|^{-\frac{1}{2}} \left[A'(x) - A'(y)\right] |A(z)|^{-\frac{1}{2}} \right\|_{\mathcal{L}(H)} \\ &\leqslant \left\| \left[A'(x) - A'(y)\right] A(w)^{-1} \right\|_{\mathcal{L}(H)}^{\frac{1}{2}} \left\| \left[A'(x) - A'(y)\right] A(z)^{-1} \right\|_{\mathcal{L}(H)}^{\frac{1}{2}} \\ &\leqslant \left\| \left[A'(x) - A'(y)\right] A(x_0)^{-1} \right\|_{\mathcal{L}(H)} \left\| A(x_0)A(w)^{-1} \right\|_{\mathcal{L}(H)}^{\frac{1}{2}} \left\| A(x_0)A(z)^{-1} \right\|_{\mathcal{L}(H)}^{\frac{1}{2}} \\ &\leqslant M_2 \left\| \left[A'(x) - A'(y)\right] A(x_0)^{-1} \right\|_{\mathcal{L}(H)}.\end{aligned}$$
   Completely analogously follows
$$\left\| |A(w)|^{-\frac{1}{2}} \left[A(x) - A(y)\right] |A(z)|^{-\frac{1}{2}} \right\|_{\mathcal{L}(H)} \leqslant M_2 \left\| \left[A(x) - A(y)\right] A(x_0)^{-1} \right\|_{\mathcal{L}(H)}.$$

Let $E_\lambda(x),\ \lambda \in \mathbb{R}$ denote the spectral family of $A(x)$ for $x \in [a,b]$ (as defined on page 354 in [Kat95]) and $u, v \in H$.
$$\begin{aligned}\int_{-\infty}^{\infty} \left\| |A(x)|^{\frac{1}{2}} (A(x) \pm it)^{-1} u \right\|_H^2 dt &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|\lambda|}{\lambda^2 + t^2} d \left\| E_\lambda(x) u \right\|_H^2 dt \\ &\overset{\text{Fubini}}{=} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|\lambda|}{\lambda^2 + t^2} dt d \left\| E_\lambda(x) u \right\|_H^2 = \pi \int_{-\infty}^{\infty} d \left\| E_\lambda(x) u \right\|_H^2 = \pi \left\| u \right\|_H^2\end{aligned}$$
This implies the key estimate in this proof:
$$\begin{aligned}&\frac{1}{\pi} \int_{-\infty}^{\infty} \left\| |A(x)|^{\frac{1}{2}} (A(x) - it)^{-1} u \right\|_H \left\| |A(x)|^{\frac{1}{2}} (A(x) + it)^{-1} v \right\|_H dt \\ &\overset{CS}{\leqslant} \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \left\| |A(x)|^{\frac{1}{2}} (A(x) - it)^{-1} u \right\|_H^2 dt \right)^{\frac{1}{2}} \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \left\| |A(x)|^{\frac{1}{2}} (A(x) + it)^{-1} v \right\|_H^2 dt \right)^{\frac{1}{2}} \\ &= \left\| u \right\|_H \left\| v \right\|_H.\end{aligned} \tag{2.1}$$
We may now estimate the norm of $J(x)$: Let $u, v \in H$.
$$\begin{aligned}&|(J(x)u, v)| = \left| \left(\frac{1}{\pi} \int_{-\infty}^{\infty} (A(x) - it)^{-1} A'(x) (A(x) - it)^{-1} dt\ u, v \right) \right| \\ &\leqslant \frac{1}{\pi} \int_{-\infty}^{\infty} \left| \left(|A(x)|^{-\frac{1}{2}} A'(x) |A(x)|^{-\frac{1}{2}} |A(x)|^{\frac{1}{2}} (A(x) - it)^{-1} u, |A(x)|^{\frac{1}{2}} (A(x) + it)^{-1} v \right) \right| dt\end{aligned}$$

$$\overset{CS}{\leqslant} \left\| |A(x)|^{-\frac{1}{2}} A'(x) |A(x)|^{-\frac{1}{2}} \right\|_{\mathcal{L}(H)} \cdot \frac{1}{\pi} \int_{-\infty}^{\infty} \left\| |A(x)|^{\frac{1}{2}} (A(x) - it)^{-1} u \right\|_H \left\| |A(x)|^{\frac{1}{2}} (A(x) + it)^{-1} v \right\|_H dt$$
$$\overset{(2.1),\ 3.}{\leqslant} \left\| A'(x) A(x)^{-1} \right\|_{\mathcal{L}(H)} \|u\|_H \|v\|_H$$

This yields the assertion

$$\|J(x)\|_{\mathcal{L}(H)} \leqslant \left\| A'(x) A(x)^{-1} \right\|_{\mathcal{L}(H)}.$$

Next, we will prove that J is continuous:

$$\begin{aligned}
\left| \Big(\big[J(x) - J(y) \big] u, v \Big) \right| &= \left| \frac{1}{\pi} \int_{-\infty}^{\infty} \Big(\big[-(A(x) - it)^{-1} A'(x) (A(x) - it)^{-1} \right. \\
&\qquad \left. + (A(y) - it)^{-1} A'(y) (A(y) - it)^{-1} \big] u, v \Big) dt \right| \\
&= \left| \frac{1}{\pi} \int_{-\infty}^{\infty} \Big(\Big[(A(y) - it)^{-1} \left[A(x) - A(y) \right] (A(x) - it)^{-1} A'(x) (A(x) - it)^{-1} \right. \\
&\qquad - (A(y) - it)^{-1} \left[A'(x) - A'(y) \right] (A(x) - it)^{-1} \\
&\qquad \left. + (A(y) - it)^{-1} A'(y) (A(y) - it)^{-1} \left[A(x) - A(y) \right] (A(x) - it)^{-1} \Big] u, v \Big) dt \right| \\
&= \left| \frac{1}{\pi} \int_{-\infty}^{\infty} \Big(\Big[\left[A(x) - A(y) \right] (A(x) - it)^{-1} A'(x) - \left[A'(x) - A'(y) \right] \right. \\
&\qquad \left. + A'(y) (A(y) - it)^{-1} \left[A(x) - A(y) \right] \Big] (A(x) - it)^{-1} u, (A(y) + it)^{-1} v \Big) dt \right| \\
&\leqslant \Bigg[\left\| |A(y)|^{-\frac{1}{2}} \left[A(x) - A(y) \right] |A(x)|^{-\frac{1}{2}} \right\|_{\mathcal{L}(H)} \Bigg\{ \left\| |A(x)|^{-\frac{1}{2}} A'(x) |A(x)|^{-\frac{1}{2}} \right\|_{\mathcal{L}(H)} \\
&\qquad + \left\| |A(y)|^{-\frac{1}{2}} A'(y) |A(y)|^{-\frac{1}{2}} \right\|_{\mathcal{L}(H)} \Bigg\} + \left\| |A(y)|^{-\frac{1}{2}} \left[A'(x) - A'(y) \right] |A(x)|^{-\frac{1}{2}} \right\|_{\mathcal{L}(H)} \Bigg] \\
&\qquad \cdot \left| \frac{1}{\pi} \int_{-\infty}^{\infty} \Big(|A(x)|^{\frac{1}{2}} (A(x) - it)^{-1} u, |A(y)|^{\frac{1}{2}} (A(y) + it)^{-1} v \Big) dt \right| \\
&\overset{(2.1),\ 4.}{\leqslant} \Big[2 M_3 M_2 \left\| \left[A(x) - A(y) \right] A(x_0)^{-1} \right\|_{\mathcal{L}(H)} \\
&\qquad + M_2 \left\| \left[A'(x) - A'(y) \right] A(x_0)^{-1} \right\|_{\mathcal{L}(H)} \Big] \|u\|_H \|v\|_H \\
&\leqslant \varepsilon \|u\|_H \|v\|_H \qquad \text{for } |x - y| < \delta
\end{aligned}$$

The ε-estimate in the last line follows since $x \mapsto A'(x) A(x_0)^{-1}$ is assumed to be continuous in norm and since $x \mapsto A(x) A(x_0)^{-1}$ is strongly continuously differentiable and thus also continuous in norm (by Lemma 3.5 on page 11 in [Kre71]).

Next, we want to show that I is differentiable in norm with derivative $J(x)$. Let $y \neq x \in [a, b]$.

$$\left| \left(\left[\frac{I(y) - I(x)}{y - x} - J(x) \right] u, v \right) \right|$$

$$
\begin{aligned}
&= \left| \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \left[\frac{(A(y)-it)^{-1} - (A(x)-it)^{-1}}{y-x} - \frac{\partial}{\partial x}(A(x)-it)^{-1} \right] dt\ u, v \right) \right| \\
&= \left| \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \left[\frac{1}{y-x} \int_x^y \frac{\partial}{\partial z}(A(z)-it)^{-1} dz - \frac{\partial}{\partial x}(A(x)-it)^{-1} \right] dt\ u, v \right) \right| \\
&= \left| \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{y-x} \int_x^y \left[\frac{\partial}{\partial z}(A(z)-it)^{-1} - \frac{\partial}{\partial x}(A(x)-it)^{-1} \right] dz\ dt\ u, v \right) \right| \\
&\overset{Fubini}{=} \left| \frac{1}{y-x} \int_x^y \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \left[\frac{\partial}{\partial z}(A(z)-it)^{-1} - \frac{\partial}{\partial x}(A(x)-it)^{-1} \right] dt\ u, v \right) dz \right| \\
&\leqslant \frac{1}{|y-x|} \left| \int_x^y \Big([J(z)-J(x)]\, u, v \Big) dz \right| \leqslant \varepsilon \frac{1}{|y-x|} \left| \int_x^y dz \right| \|u\|_H \|v\|_H \\
&= \varepsilon \|u\|_H \|v\|_H \qquad \text{for } |x-y| < \delta
\end{aligned}
$$

□

Lemma 2.2.2. *If the family of self-adjoint operators $x \mapsto S(x) \in \mathcal{L}(H_1, H)$ is strongly differentiable, $x \mapsto S'(x)S(s_0)^{-1}$ is norm continuous and $\ker S(x) = \varnothing$ for all $x \in I$, then the families of spectral projections $Q_>$ and $Q_<$ of S corresponding to the intervals $[C_2, \infty)$ and $(-\infty, -C_2]$, respectively, are continuously differentiable in norm and*

$$\left\| Q'_>(x) \right\|_{\mathcal{L}(H)} + \left\| Q'_<(x) \right\|_{\mathcal{L}(H)} \leqslant C\alpha(x).$$

If S also satisfies the HO^2 property, then the assumptions on the spectral projections (Definition 2.1.1, 3.) follow automatically.

Proof. We define as in Lemma VI.5.6 on page 359 in [Kat95]

$$U(x) := \frac{1}{\pi} \int_{-\infty}^{\infty} (S(x) - it)^{-1}\, dt.$$

From the same lemma follows

$$U(x)u = \begin{cases} u, & u \in Q_>(x)H \\ -u, & u \in Q_<(x)H \end{cases}.$$

Based on these observations, the projections are given by

$$Q_>(x) = \frac{1}{2}\left(U^2(x) + U(x)\right) \quad \text{and} \quad Q_<(x) = \frac{1}{2}\left(U^2(x) - U(x)\right).$$

From Lemma 2.2.1 follows the continuous differentiability of $U_\pm$ in norm and

$$\left\| U'(x) \right\| \leqslant \left\| S'(x)S(x)^{-1} \right\|_{\mathcal{L}(H)} = \alpha(x).$$

If S also satisfies the HO^2 property, this implies

$$\int_0^{s_0} x^\beta \left(\left\| Q'_>(x) \right\|^2_{\mathcal{L}(H)} + \left\| Q'_<(x) \right\|^2_{\mathcal{L}(H)} \right) dx \leqslant 2C^2 \int_0^{s_0} x^\beta \alpha^2(x) dx \overset{HO^2}{<} \infty.$$

□

3 Proof of the index theorem

3.1 The case of constant spectral projections

In this section we look at the horn operator

$$\mathrm{T}_{\mathrm{tot}} = \left(\frac{\partial}{\partial x} + x^{-\beta} S(x)\right) \oplus \left(\frac{\partial}{\partial x} + x^{-1} \widetilde{S}\right)$$

and assume that $Q_>(x)$ and $Q_<(x)$ are independent of x. This implies that the decomposition $H = Q_< H \oplus Q_> H$ does not depend on x. In this case, a parametrix P will be constructed and its properties proved.

Lemma 3.1.1. [1] *Let $0 < \delta \leqslant s_0$ and*

$$\Delta_\delta := \left\{(x,y) \in \mathbb{R}^2 \mid \delta \leqslant y \leqslant x \leqslant s_0\right\}.$$

For $e \in Q_> H_1$, $(x,y) \in \Delta_\delta$, the unique solution of the initial value problem

$$\begin{aligned} \left(\frac{\partial}{\partial x} + x^{-\beta} S_>(x)\right) u(x) &= 0, \quad x > y \\ u(y) &= e, \end{aligned}$$

is given by $u(x) = W_>(x,y)e$. The solution operators have the following properties:

1. *$W_>(x,y)$ is uniformly bounded and strongly continuous in Δ_δ.*
2. *$W_>(x,z)W_>(z,y) = W_>(x,y)$, $W_>(x,x) = \mathbb{1}$, $\forall\ \delta \leqslant y \leqslant z \leqslant x \leqslant s_0$.*
3. *$W_>(x,y)(Q_> H_1) \subset Q_> H_1$ and the operator $S_>(x)W_>(x,y)S_>^{-1}(y)$ is bounded and strongly continuous in Δ_δ.*
4. *$W_>(x,y)$ is strongly continuously differentiable in $Q_> H_1$ relative to x and y and*

$$\frac{\partial}{\partial x} W_>(x,y) = -x^{-\beta} S_>(x) W_>(x,y) \quad \text{and} \quad \frac{\partial}{\partial y} W_>(x,y) = W_>(x,y) x^{-\beta} S_>(y). \tag{3.1}$$

Proof. Essentially, this lemma is an application of Theorem 3.11 in [Kre71] (page 208). Unfortunately, the theorem is not applicable in the form given there. The adjusted statement is formulated and completely proved as Theorem 4.2.6 on page 69 below.

In order to apply this theorem, we have to show that the operator family

$$A(x) := -(x+\delta)^{-\beta} S_>(x+\delta)$$

satisfies the following two assumptions:

[1] The content of this lemma is similar to Lemma 3.1 in [Brü92], where it is stated for the cone case.

1. $A(x)$ is strongly continuously differentiable for $x \in [0, s_0 - \delta]$:
   Let $e \in H_1$, then

$$\begin{aligned}(A(x)e)' &= (-(x+\delta)^{-\beta} S_>(x+\delta)e)' \\ &= \beta(x+\delta)^{-\beta-1}(S_>(x+\delta)e) - (x+\delta)^{-\beta}(S_>(x+\delta)e)'.\end{aligned}$$

   Since $S_>$ is strongly continuously differentiable in $[\delta, s_0]$ and thus also strongly continuous, the right side is strongly continuous.

2. $A(x)$ has a bounded inverse and $\left\|(A(x) - \lambda \mathbb{1})^{-1}\right\|_{\mathcal{L}(Q_> H)} \leqslant \lambda^{-1},\ \forall \lambda > 0:$
   It suffices to find an $\varepsilon > 0$, such that $\left\|(A(x) - \lambda \mathbb{1})^{-1}\right\|_{\mathcal{L}(Q_> H)} \leqslant (\varepsilon + \lambda)^{-1}$ for $\lambda \geqslant 0$:

$$\begin{aligned}\left\|(A(x) - \lambda \mathbb{1})^{-1}\right\|_{\mathcal{L}(Q_> H)} &= \left\|\left(-(x+\delta)^{-\beta} S_>(x+\delta) - \lambda \mathbb{1}\right)^{-1}\right\|_{\mathcal{L}(Q_> H)} \\ &\leqslant \frac{1}{\operatorname{dist}(\operatorname{spec}(-(x+\delta)^{-\beta} S_>(x+\delta)), \lambda)} \\ &\leqslant \frac{1}{(x+\delta)^{-\beta} C_2 + \lambda} \leqslant \frac{1}{s_0^{-\beta} C_2 + \lambda} \overset{\varepsilon := s_0^{-\beta} C_2}{=} \frac{1}{\varepsilon + \lambda}\end{aligned}$$

□

To ease notation, we define

$$F(x,y) := \int_y^x t^{-\beta} dt = \frac{1}{\beta - 1}\left(y^{1-\beta} - x^{1-\beta}\right).$$

Then $\frac{\partial}{\partial x} F(x,y) = x^{-\beta}$.

The following technical lemma plays a central role in the proof of the Fredholm property of operators with horn singularities (Its twin Lemma 3.3.5 will play the central role in the calculation of the Fredholm index).

Lemma 3.1.2. [2] *Let $d \in (0, C_2)$. For every $0 < \varepsilon < \frac{1}{2}$ in the HO^2 case and every $0 < \varepsilon < 1$ in the HO$^\infty$ case exists a constant $C > 0$, such that*

$$\|W_>(x,y)\|_{\mathcal{L}(Q_> H)} \leqslant C e^{-dF(x,y)}\left(F(x,y)^{-\varepsilon} + 1\right), \quad \forall\ 0 < y < x \leqslant s_0.$$

Proof. By assumption $\operatorname{spec} S_>(x) \subset [C_2, \infty)$ for all $x \in I$. We define

$$c(t) := d + |t| + it, \quad t \in \mathbb{R}.$$

Thus, we can define the contour integral

$$\widetilde{W}_>(x,y) := \frac{1}{2\pi i}\int_c e^{-\zeta F(x,y)}(S_>(x) - \zeta)^{-1} d\zeta, \quad \text{for } 0 \leqslant y < x \leqslant s_0$$

[2] This lemma is related to Lemma 3.2 in [Brü92], where a similar statement is proved for the cone case. The transition to the horn changed the estimates quite a bit. Furthermore, it took a lot of work to remove the assumption on the smallness of C_4. That was necessary to bridge a gap in [Brü92].

as an approximation of $W_>$.

Differentiation with regard to x yields

$$\begin{aligned}
\frac{\partial}{\partial x}\widetilde{W}_>(x,y) = &\frac{\partial}{\partial x}\left(\frac{1}{2\pi i}\int_c e^{-\zeta F(x,y)}(S_>(x)-\zeta)^{-1}d\zeta\right)\\
=&\frac{1}{2\pi i}\int_c(-\zeta x^{-\beta})e^{-\zeta F(x,y)}(S_>(x)-\zeta)^{-1}d\zeta\\
&+\underbrace{\frac{1}{2\pi i}\int_c e^{-\zeta F(x,y)}(-1)(S_>(x)-\zeta)^{-1}(S_>)'(x)(S_>(x)-\zeta)^{-1}d\zeta}_{=:\widetilde{R}(x,y)}\\
=&\frac{1}{2\pi i}\int_c x^{-\beta}(-S_>(x)+(S_>(x)-\zeta))e^{-\zeta F(x,y)}(S_>(x)-\zeta)^{-1}d\zeta+\widetilde{R}(x,y)\\
=&-x^{-\beta}S_>(x)\frac{1}{2\pi i}\int_c e^{-\zeta F(x,y)}(S_>(x)-\zeta)^{-1}d\zeta + x^{-\beta}\underbrace{\frac{1}{2\pi i}\int_c e^{-\zeta F(x,y)}d\zeta}_{\overset{\text{Cauchy}}{=}0}+\widetilde{R}(x,y)\\
=&-x^{-\beta}S_>(x)\widetilde{W}_>(x,y)+\widetilde{R}(x,y).
\end{aligned}$$

Let $f \in C_0^\infty(I, Q_>H_1)$ and define

$$\widetilde{u}(x) := \int_\delta^x \widetilde{W}_>(x,y)f(y)dy,$$

then

$$\begin{aligned}
\widetilde{u}'(x) &= \underbrace{\widetilde{W}_>(x,x)}_{=\mathbb{1}}f(x)+\int_\delta^x\frac{\partial}{\partial x}\widetilde{W}_>(x,y)f(y)dy\\
&= f(x)-x^{-\beta}S_>(x)\widetilde{u}(x)+\int_\delta^x\widetilde{R}(x,y)f(y)dy\\
&=: -x^{-\beta}S_>(x)\widetilde{u}(x)+g(x).
\end{aligned}$$

The function g is continuous in $[\delta, s_0]$ since $f \in C_0^\infty(I, Q_>H_1)$ and $\widetilde{R}(x,y) : Q_>H \to Q_>H$ is strongly continuous in x and y. Furthermore, $\widetilde{u}(x) \in Q_>H_1$ for all x and $\widetilde{u}$ satisfies the inhomogeneous differential equation

$$\left(\frac{\partial}{\partial x}+x^{-\beta}S_>(x)\right)\widetilde{u}(x) = g(x), \qquad \widetilde{u}(\delta)=0.$$

It follows from Theorem 3.1 on page 195 in [Kre71] that

$$\widetilde{u}(x) = \int_\delta^x W_>(x,y)g(y)dy.$$

This gives

$$\int_\delta^x \widetilde{W}_>(x,y)f(y)dy = \int_\delta^x W_>(x,y)f(y)dy + \int_\delta^x W_>(x,y)\int_\delta^y\widetilde{R}(y,z)f(z)dzdy$$

or

$$\begin{aligned}\int_\delta^x W_>(x,y)f(y)dy &= \int_\delta^x \widetilde{W}_>(x,y)f(y)dy - \int_\delta^x W_>(x,y)\int_\delta^y \widetilde{R}(y,z)f(z)dzdy \\ &= \int_\delta^x \widetilde{W}_>(x,y)f(y)dy - \int_\delta^x \underbrace{\int_z^x W_>(x,y)\widetilde{R}(y,z)dy}_{=:W_>\tilde{*}\widetilde{R}(x,z)} f(z)dz.\end{aligned}$$

Since $C_0^\infty(I,Q_>H_1)$ is dense in $L^2(I,Q_>H)$, we conclude that

$$W_>(x,y)=\widetilde{W}_>(x,y)-\left(W_>\tilde{*}\widetilde{R}\right)(x,y),\quad \delta\leqslant y<x\leqslant s_0.$$

From this it follows by induction that for all $N\in\mathbb{N}$

$$W_>(x,y)=\sum_{j=0}^N(-1)^j\left(\widetilde{W}_>\tilde{*}\widetilde{R}^j\right)(x,y)+(-1)^{N+1}\left(W_>\tilde{*}\widetilde{R}^{N+1}\right)(x,y), \tag{3.2}$$

where $\widetilde{R}^j(x,y):=(\underbrace{\widetilde{R}\tilde{*}\cdots\tilde{*}\widetilde{R}}_{j-\text{times}})(x,y)$.

For the rest of the lemma $\|\cdot\|$ will always mean $\|\cdot\|_{\mathcal{L}(Q_>H)}$. We want to calculate the norms of the terms on the right side of Formula (3.2) and start by proving the following estimates

$$\left\|\widetilde{W}_>(x,y)\right\|\leqslant K_1e^{-dF(x,y)}F(x,y)^{-\varepsilon}\qquad\text{and} \tag{3.3}$$

$$\left\|\widetilde{R}(x,y)\right\|\leqslant K_2e^{-dF(x,y)}F(x,y)^{-\varepsilon}\alpha(x), \tag{3.4}$$

where K_1 and K_2 can be chosen uniformly for $\delta\leqslant y<x\leqslant s_0$.

The first estimate follows from the calculation

$$\begin{aligned}\left\|\widetilde{W}_>(x,y)\right\| &= \left\|\frac{1}{2\pi i}\int_c e^{-\zeta F(x,y)}(S_>(x)-\zeta)^{-1}d\zeta\right\| \\ &\leqslant \frac{1}{2\pi}\int_c\left|e^{-\zeta F(x,y)}\right|\left\|(S_>(x)-\zeta)^{-1}\right\|d\zeta \\ &\leqslant \widetilde{\widetilde{K_1}}e^{-dF(x,y)}\int_c\left|e^{-(\zeta-d)F(x,y)}\right||\zeta|^{-1}d\zeta, && \widetilde{\widetilde{K_1}}=\widetilde{\widetilde{K_1}}(c,C_2) \\ &\leqslant \widetilde{K_1}e^{-dF(x,y)}F(x,y)^{-\varepsilon}\int_c|\zeta-d|^{-\varepsilon}|\zeta|^{-1}d\zeta, && \widetilde{K_1}=\widetilde{K_1}(c,C_2,\varepsilon) \\ &\leqslant K_1e^{-dF(x,y)}F(x,y)^{-\varepsilon}, && K_1=K_1(c,C_2,\varepsilon).\end{aligned}$$

The second estimate is implied by

$$\begin{aligned}\left\|\widetilde{R}(x,y)\right\| &= \frac{1}{2\pi}\int_c\left|e^{-\zeta F(x,y)}\right|\left\|(S_>(x)-\zeta)^{-1}\right\|\left\|(S_>)'(x)(S_>(x)-\zeta)^{-1}\right\|d\zeta \\ &\leqslant \widetilde{K_2}e^{-dF(x,y)}\left(F(x,y)\right)^{-\varepsilon}\left\|(S_>)'(x)S_>(x)^{-1}\right\| \\ &\leqslant K_2e^{-dF(x,y)}\left(F(x,y)\right)^{-\varepsilon}\alpha(x), && K_2=K_2(c,C_2,\varepsilon).\end{aligned}$$

Let

$$K:=\max\{K_1,K_2\}.$$

HO^2 case:

The following estimates are different in the HO$^\infty$ and the HO^2 case. We start by proving the lemma in the HO^2 case. Remember, in this case

$$C_4(s) = \left(\int_0^s x^\beta \alpha^2(x)dx\right)^{\frac{1}{2}} < \infty.$$

Assertion 1:

$$\left\|\widetilde{R}^j(x,y)\right\| \leqslant K\left[F(x,y)^{\frac{1}{2}-\varepsilon}C_x\right]^{j-1} e^{-dF(x,y)}F(x,y)^{-\varepsilon}\alpha(x), \quad \forall\ 0 \leqslant y < x \leqslant s_0, \tag{3.5}$$

with

$$C_x := KB(1-2\varepsilon, 1-2\varepsilon)^{\frac{1}{2}}C_4(x) \xrightarrow{x\to 0} 0,$$

where

$$B(a,b) := \int_0^1 t^{a-1}(1-t)^{b-1}dt, \quad \mathrm{Re}(a) > 0,\ \mathrm{Re}(b) > 0$$

denotes *Euler's beta function.*

Assertion 1 is proved by induction over j. The case $j = 1$ is simply Estimate (3.4). Let us assume that the estimate is proved for a $j \in \mathbb{N}$.

$$\begin{aligned}
&\left\|\widetilde{R}^{j+1}(x,y)\right\| = \left\|\int_y^x \widetilde{R}(x,z)\widetilde{R}^j(z,y)dz\right\| \leqslant \int_y^x \left\|\widetilde{R}(x,z)\right\| \left\|\widetilde{R}^j(z,y)\right\| dz \\
&\leqslant K^2\left[F(x,y)^{\frac{1}{2}-\varepsilon}C_x\right]^{j-1} \int_y^x e^{-d(F(x,z)+F(z,y))}F(x,z)^{-\varepsilon}\alpha(x)F(z,y)^{-\varepsilon}\alpha(z)dz \\
&\leqslant K^2\left[F(x,y)^{\frac{1}{2}-\varepsilon}C_x\right]^{j-1} e^{-dF(x,y)}\alpha(x)\left(\int_y^x F(x,z)^{-2\varepsilon}F(z,y)^{-2\varepsilon}\frac{dz}{z^\beta}\right)^{\frac{1}{2}}\left(\int_y^x z^\beta\alpha^2(z)dz\right)^{\frac{1}{2}} \\
&\leqslant K^2\left[F(x,y)^{\frac{1}{2}-\varepsilon}C_x\right]^{j-1} e^{-dF(x,y)}\alpha(x)\left(\int_{F(y,y)=0}^{F(x,y)} (F(x,y)-s)^{-2\varepsilon}\, s^{-2\varepsilon}ds\right)^{\frac{1}{2}} C_4(x) \\
&= K^2\left[F(x,y)^{\frac{1}{2}-\varepsilon}C_x\right]^{j-1} e^{-dF(x,y)}\alpha(x)F(x,y)^{\frac{1}{2}-2\varepsilon}\left(\int_0^1 (1-t)^{-2\varepsilon}t^{-2\varepsilon}dt\right)^{\frac{1}{2}} C_4(x) \\
&\leqslant K\left[F(x,y)^{\frac{1}{2}-\varepsilon}C_x\right]^{j-1}\left[F(x,y)^{\frac{1}{2}-\varepsilon}KB(1-2\varepsilon,1-2\varepsilon)^{\frac{1}{2}}C_4(x)\right] e^{-dF(x,y)}F(x,y)^{-\varepsilon}\alpha(x) \\
&= K\left[F(x,y)^{\frac{1}{2}-\varepsilon}C_x\right]^{j} e^{-dF(x,y)}F(x,y)^{-\varepsilon}\alpha(x)
\end{aligned}$$

If the calculation is repeated once more leaving out $\alpha(x)$ and applying (3.3) instead of (3.4), we get the result

$$\left\|\left(\widetilde{W}_{>}\tilde{*}\widetilde{R}^j\right)(x,y)\right\| \leqslant K\left[F(x,y)^{\frac{1}{2}-\varepsilon}C_x\right]^{j} e^{-dF(x,y)}F(x,y)^{-\varepsilon}, \tag{3.6}$$

$$\forall\ 0 < y < x \leqslant s_0,\ j \in \mathbb{N}\cup\{0\}.$$

Thus, we have estimated the norms of the terms appearing in the sum on the right-hand side of Equation (3.2). We proceed by calculating the norm of the rest term. Lemma 3.1.1, 1. states that $W_>(x,y)$ (without tilde) is uniformly bounded for $\delta \leqslant y \leqslant x \leqslant s_0$ for any $0 < \delta \leqslant s_0$. Therefore, it exists a constant C_δ, such that $\|W_>(x,y)\| \leqslant C_\delta$ and

$$\begin{aligned}
\left\| \left(W_> \tilde{*} \widetilde{R}^{N+1} \right)(x,y) \right\| &\leqslant \int_y^x \|W_>(x,z)\| \left\| \widetilde{R}^{N+1}(z,y) \right\| dz \\
&\overset{(3.5)}{\leqslant} C_\delta K \left[F(x,y)^{\frac{1}{2}-\varepsilon} C_x \right]^N \int_y^x e^{-dF(z,y)} (F(z,y))^{-\varepsilon} \alpha(z) dz \\
&\leqslant C_\delta K \left[F(x,y)^{\frac{1}{2}-\varepsilon} C_x \right]^N \left(\int_y^x e^{-2dF(z,y)} (F(z,y))^{-2\varepsilon} z^{-\beta} dz \right)^{\frac{1}{2}} \left(\int_y^x z^\beta \alpha^2(z) dz \right)^{\frac{1}{2}} \\
&\leqslant C_\delta K \left[F(x,y)^{\frac{1}{2}-\varepsilon} C_x \right]^N \underbrace{\left(\int_0^\infty e^{-2dw} w^{-2\varepsilon} dw \right)^{\frac{1}{2}}}_{=:K_3 \overset{0<\varepsilon<\frac{1}{2}}{<} \infty} C_4(s_0) \\
&\leqslant (C_\delta K K_3 C_4(s_0)) \left[F(x,y)^{\frac{1}{2}-\varepsilon} C_x \right]^N, \qquad \forall\ \delta \leqslant y < x \leqslant s_0.
\end{aligned}$$

For all $0 < y < x \leqslant s_0$ with

$$F(x,y)^{\frac{1}{2}-\varepsilon} C_x < 1$$

the norm of the rest term converges to zero and thus, the series implied by Formula (3.2) converges. We can estimate its norm:

$$\begin{aligned}
\|W_>(x,y)\| &\overset{(3.2)}{\leqslant} \sum_{j=0}^\infty \left\| \left(\widetilde{W}_> \tilde{*} \widetilde{R}^j \right)(x,y) \right\| \\
&\overset{(3.6)}{\leqslant} K e^{-dF(x,y)} (F(x,y))^{-\varepsilon} \sum_{j=0}^\infty \left[F(x,y)^{\frac{1}{2}-\varepsilon} C_x \right]^j \\
&= \frac{K}{1 - F(x,y)^{\frac{1}{2}-\varepsilon} C_x} e^{-dF(x,y)} F(x,y)^{-\varepsilon}, \\
&\quad \forall\ 0 < y < x \leqslant s_0 \text{ with } F(x,y)^{\frac{1}{2}-\varepsilon} C_x < 1.
\end{aligned} \tag{3.7}$$

Assertion 2 It exists a $C > 0$, such that

$$\|W_>(x,y)\| \leqslant C e^{-dF(x,y)} \left(F(x,y)^{-\varepsilon} + 1 \right), \qquad \forall\ 0 < y < x \leqslant s_0.$$

Choose an $A > 0$ and $s_3 > 0$ small enough, such that

$$A^{\frac{1}{2}-\varepsilon} C_{s_3} < 1 \text{ and } \frac{K \left(\frac{A}{2} \right)^{-\varepsilon}}{1 - A^{\frac{1}{2}-\varepsilon} C_{s_3}} \leqslant 1. \tag{3.8}$$

For instance, this can be achieved by choosing $A = (4K)^{\frac{1}{\varepsilon}}$ and s_3 small enough, such that $C_{s_3} \leqslant \frac{A^{\varepsilon-\frac{1}{2}}}{2}$. Next, choose $0 < B \leqslant A$, such that

$$B^{\frac{1}{2}-\varepsilon} C_{s_0} < 1. \tag{3.9}$$

Note that A, s_3 and B just depend on K, s_0 and ε.

Case 1: $F(x,y) \leqslant B$

$$\|W_>(x,y)\| \overset{(3.7)}{\leqslant} \frac{K}{1 - F(x,y)^{\frac{1}{2}-\varepsilon} C_x} e^{-dF(x,y)} F(x,y)^{-\varepsilon} \leqslant \frac{K}{1 - B^{\frac{1}{2}-\varepsilon} C_{s_0}} e^{-dF(x,y)} F(x,y)^{-\varepsilon}$$

Case 2: $B < F(x,y) \leqslant A + B + F(s_0, s_3)$

Let $N := \left\lceil \frac{A+B+F(s_0,s_3)}{B} \right\rceil$ and choose $n \in (1, \ldots, N-1)$, such that $nB < F(x,y) \leqslant (n+1)B$. We define

$$z_j := \left(y^{-\beta+1} - j(\beta - 1) \frac{F(x,y)}{n+1} \right)^{\frac{1}{1-\beta}}, \quad j = 0, 1, \ldots, n+1.$$

Then follows

$$\begin{aligned}
z_0 &= y, \\
z_{n+1} &= \left(y^{-\beta+1} - (\beta-1) F(x,y) \right)^{\frac{1}{1-\beta}} = \left(y^{-\beta+1} - \left(y^{-\beta+1} - x^{-\beta+1} \right) \right)^{\frac{1}{1-\beta}} = x \\
F(z_{j+1}, z_j) &= \int_{z_j}^{z_{j+1}} t^{-\beta} dt = \frac{1}{\beta - 1} \left(z_j^{-\beta+1} - z_{j+1}^{-\beta+1} \right) \qquad\qquad j = 0, \ldots, n \\
&= \frac{1}{\beta-1} \left(y^{-\beta+1} - j(\beta-1) \frac{F(x,y)}{n+1} - y^{-\beta+1} + (j+1)(\beta-1) \frac{F(x,y)}{n+1} \right) = \frac{F(x,y)}{n+1}.
\end{aligned}$$

By the choice of n this implies

$$\frac{n}{n+1} B < F(z_{j+1}, z_j) \leqslant B$$

and in particular, $F(z_{j+1}, z_j)^{\frac{1}{2}-\varepsilon} C_{z_{j+1}} \leqslant B^{\frac{1}{2}-\varepsilon} C_{s_0} \overset{(3.9)}{<} 1$. Using the product rule of solutions (see Lemma 3.1.1, 2.), we estimate

$$\begin{aligned}
\|W_>(x,y)\| &\leqslant \prod_{j=0}^{n} \|W_>(z_{j+1}, z_j)\| \overset{(3.7)}{\leqslant} \prod_{j=0}^{n} \frac{K F(z_{j+1}, z_j)^{-\varepsilon}}{1 - F(z_{j+1}, z_j)^{\frac{1}{2}-\varepsilon} C_{z_{j+1}}} e^{-dF(z_{j+1}, z_j)} \\
&\leqslant \left[\frac{K \left(\frac{n}{n+1} B \right)^{-\varepsilon}}{1 - B^{\frac{1}{2}-\varepsilon} C_{s_0}} \right]^{n+1} e^{-dF(x,y)} \leqslant \max \left\{ \left[\frac{K \left(\frac{B}{2} \right)^{-\varepsilon}}{1 - B^{\frac{1}{2}-\varepsilon} C_{s_0}} \right]^{N}, 1 \right\} e^{-dF(x,y)}.
\end{aligned}$$

Case 3: $0 < y < x \leqslant s_3$ **and** $F(x,y) > A$

(The case $F(x,y) \leqslant A$ is covered by case 1 and case 2 above.) Choose $m \in \mathbb{N}$, such that $mA < F(x,y) \leqslant (m+1)A$. Again, we define intermediate points

$$v_j := \left(y^{-\beta+1} - j(\beta-1)\frac{F(x,y)}{m+1}\right)^{\frac{1}{1-\beta}}, \quad j = 0, 1, \ldots, m+1.$$

This implies

$$\begin{aligned}
&v_0 = y,\ v_{m+1} = x,\ F(v_{j+1}, v_j) = \frac{F(x,y)}{m+1}, \\
&\frac{m}{m+1}A < F(v_{j+1}, v_j) \leqslant A \text{ and } F(v_{j+1}, v_j)^{\frac{1}{2}-\varepsilon} C_{v_{j+1}} \leqslant A^{\frac{1}{2}-\varepsilon} C_{s_3} < 1.
\end{aligned}$$

Using the product rule of solutions again, we estimate

$$\begin{aligned}
\|W_>(x,y)\| &\leqslant \prod_{j=0}^{m} \|W_>(v_{j+1}, v_j)\| \overset{(3.7)}{\leqslant} \prod_{j=0}^{m} \frac{KF(v_{j+1}, v_j)^{-\varepsilon}}{1 - F(v_{j+1}, v_j)^{\frac{1}{2}-\varepsilon} C_{v_{j+1}}} e^{-dF(v_{j+1}, v_j)} \\
&\leqslant \left[\frac{K\left(\frac{m}{m+1}A\right)^{-\varepsilon}}{1 - A^{\frac{1}{2}-\varepsilon} C_{s_3}}\right]^{m+1} e^{-dF(x,y)} \leqslant \left[\frac{K\left(\frac{A}{2}\right)^{-\varepsilon}}{1 - A^{\frac{1}{2}-\varepsilon} C_{s_3}}\right]^{m+1} e^{-dF(x,y)} \overset{(3.8)}{\leqslant} e^{-dF(x,y)}.
\end{aligned}$$

Case 4: $0 < y < s_3 < x \leqslant s_0$ **and** $F(s_3, y) > A + B$

(If $F(s_3, y) \leqslant A + B$, then $F(x,y) = F(x, s_3) + F(s_3, y) \leqslant F(s_0, s_3) + A + B$. This is either case 1 or case 2. If $s_3 \leqslant y < x \leqslant s_0$, then $F(x,y) \leqslant F(s_0, s_3)$. This is case 1 or 2 again.)

We define one intermediate point in order to reduce case 4 to case 2 and 3.

$$\begin{aligned}
z &:= \left(s_3^{-\beta+1} + (\beta-1)B\right)^{\frac{1}{1-\beta}} < s_3 \\
F(s_3, z) &= \frac{1}{\beta-1}\left(z^{-\beta+1} - s_3^{-\beta+1}\right) = \frac{1}{\beta-1}\left(s_3^{-\beta+1} + (\beta-1)B - s_3^{-\beta+1}\right) = B \\
F(x,z) &= F(x, s_3) + F(s_3, z) = F(x, s_3) + B \ \Rightarrow\ B < F(x,z) \leqslant F(s_0, s_3) + B \quad \text{(Case 2)} \\
F(z,y) &= F(s_3, y) - F(s_3, z) > A + B - B = A \quad \text{(Case 3)}
\end{aligned}$$

This implies

$$\begin{aligned}
\|W_>(x,y)\| &\leqslant \|W_>(x,z)\| \|W_>(z,y)\| \\
&\leqslant \max\left\{\left[\frac{K\left(\frac{B}{2}\right)^{-\varepsilon}}{1 - B^{\frac{1}{2}-\varepsilon} C_{s_0}}\right]^N, 1\right\} e^{-dF(x,z)} e^{-dF(z,y)} \\
&= \max\left\{\left[\frac{K\left(\frac{B}{2}\right)^{-\varepsilon}}{1 - B^{\frac{1}{2}-\varepsilon} C_{s_0}}\right]^N, 1\right\} e^{-dF(x,y)}.
\end{aligned}$$

Combining the estimates gained in the four cases above, we prove Assertion 2 and thus the lemma:

$$\|W_>(x,y)\| \leqslant \max\left\{\frac{K}{1-B^{\frac{1}{2}-\varepsilon}C_{s_0}}, \left(\frac{K\left(\frac{B}{2}\right)^{-\varepsilon}}{1-B^{\frac{1}{2}-\varepsilon}C_{s_0}}\right)^N, 1\right\} \quad e^{-dF(x,y)}\left(F(x,y)^{-\varepsilon}+1\right)$$

$$\forall\ 0<y<x\leqslant s_0.$$

HO$^\infty$ case:

In this case

$$C_4(s) = \sup_{z\in(0,s]} z^\beta \alpha(z).$$

Assertion 1:

$$\left\|\widetilde{R}^j(x,y)\right\| \leqslant K(F(x,y)^{1-\varepsilon}C_x)^{j-1}e^{-dF(x,y)}F(x,y)^{-\varepsilon}\alpha(x), \quad \forall\ 0\leqslant y<x\leqslant s_0,$$

with

$$C_x := KB(1-\varepsilon,1-\varepsilon)C_4(x) \stackrel{x\to 0}{\longrightarrow} 0,$$

where B denotes *Euler's beta function* again.

Assertion 1 is again proved by induction over j, and the case $j=1$ is the estimate (3.4). Let us assume that the estimate is proved for a $j\in\mathbb{N}$.

$$\begin{aligned}
\left\|\widetilde{R}^{j+1}(x,y)\right\| &\leqslant K^2\left[F(x,y)^{1-\varepsilon}C_x\right]^{j-1}\int_y^x e^{-d(F(x,z)+F(z,y))}F(x,z)^{-\varepsilon}\alpha(x)F(z,y)^{-\varepsilon}\alpha(z)dz \\
&\leqslant K^2\left[F(x,y)^{1-\varepsilon}C_x\right]^{j-1}e^{-dF(x,y)}\alpha(x)\left(\int_y^x F(x,z)^{-\varepsilon}F(z,y)^{-\varepsilon}\frac{dz}{z^\beta}\right)\left(\sup_{z\in(0,x]} z^\beta\alpha(z)\right) \\
&\leqslant K^2\left[F(x,y)^{1-\varepsilon}C_x\right]^{j-1}e^{-dF(x,y)}\alpha(x)\left(\int_{F(y,y)=0}^{F(x,y)}(F(x,y)-s)^{-\varepsilon}s^{-\varepsilon}ds\right)C_4(x) \\
&= K^2\left[F(x,y)^{1-\varepsilon}C_x\right]^{j-1}e^{-dF(x,y)}\alpha(x)F(x,y)^{1-2\varepsilon}\left(\int_0^1(1-t)^{-\varepsilon}t^{-\varepsilon}dt\right)C_4(x) \\
&\leqslant K\left[F(x,y)^{1-\varepsilon}C_x\right]^{j-1}\left(F(x,y)^{1-\varepsilon}KB(1-\varepsilon,1-\varepsilon)C_4(x)\right)e^{-dF(x,y)}F(x,y)^{-\varepsilon}\alpha(x) \\
&= K\left[F(x,y)^{1-\varepsilon}C_x\right]^{j}e^{-dF(x,y)}F(x,y)^{-\varepsilon}\alpha(x)
\end{aligned}$$

The rest of the proof is the same as in the HO^2 case if $\frac{1}{2}-\varepsilon$ is replaced by $1-\varepsilon$. □

Let $V_>(x,y)$ be the solution operator corresponding to the equation

$$\left(\frac{\partial}{\partial x}-\frac{1}{x^\beta}S_<(x)\right)u(x)=0.$$

Since $-x^{-\beta}S_<(x)$ is a positive operator, $V_>$ has the same properties as $W_>$, i.e. it also satisfies Lemma 3.1.1. Let $e \in Q_<H$, $f \in Q_<H_1$ and $(x,y) \in \Delta_\delta$:

$$\begin{aligned}
&\left\langle \left(\frac{\partial}{\partial x} + x^{-\beta} S_<(x) \right) V_>(y,x)^* e, f \right\rangle_{Q_<H} \\
&\qquad = \frac{\partial}{\partial x} \langle e, V_>(y,x) f \rangle_{Q_<H} + \left\langle e, V_>(y,x) x^{-\beta} S_<(x) f \right\rangle_{Q_<H} \\
&\qquad = \left\langle e, \left(\frac{\partial}{\partial x} V_>(y,x) - V_>(y,x) \left[-x^{-\beta} S_<(x) \right] \right) f \right\rangle_{Q_<H} \overset{3.1.1,4}{=} 0.
\end{aligned}$$

We can apply Lemma 3.1.2 and get the same estimate for $W_<(x,y) := V_>(y,x)^*$ as before for $W_>(x,y)$ since the adjoint operator has the same norm:

Corollary 3.1.3. [3] *Let $d \in (0, C_2)$. For every $0 < \varepsilon < \frac{1}{2}$ in the HO^2 case and every $0 < \varepsilon < 1$ in the HO$^\infty$ case exists a $C > 0$, such that*

$$\|W_<(x,y)\|_{\mathcal{L}(Q_<H)} \leqslant C e^{-dF(y,x)} \left(F(y,x)^{-\varepsilon} + 1 \right) \qquad 0 < x < y \leqslant s_0.$$

Remark: Pay attention to the interchanged variables x and y.

On page 662 in [BS88] the operators

$$\begin{aligned}
P_{0,\lambda} f(x) &:= x^{-\lambda} \int_0^x y^\lambda f(y) dy, && \text{for } \lambda > -\frac{1}{2} \\
P_{1,\lambda} f(x) &:= x^{-\lambda} \int_{s_0}^x y^\lambda f(y) dy, && \text{for } \lambda < \frac{1}{2}
\end{aligned}$$

for all $f \in L^2(0, s_0)$ are defined. Since $\widetilde{S}$ is a symmetric matrix, $\widetilde{H}$ can be decomposed into the finitely many eigenspaces of $\widetilde{S}$

$$\widetilde{H} = \bigoplus_{\lambda \in \operatorname{spec} \widetilde{S}} \ker(\widetilde{S} - \lambda).$$

Definition 3.1.4. We define the operators

$$\begin{aligned}
P_> f(x) &:= \int_0^x W_>(x,y) f(y) dy, && f \in L^2((0,s_0], Q_>H) \\
P_< f(x) &:= -\int_x^{s_0} W_<(x,y) f(y) dy, && f \in L^2((0,s_0], Q_<H) \\
\widetilde{P} f(x) &:= \bigoplus_{\substack{\lambda \in \operatorname{spec} \widetilde{S} \\ \lambda \leqslant -\frac{1}{2}}} P_{1,\lambda} f(x) \oplus \bigoplus_{\substack{\lambda \in \operatorname{spec} \widetilde{S} \\ \lambda > -\frac{1}{2}}} P_{0,\lambda} f(x) && f \in L^2((0,s_0], \widetilde{H})
\end{aligned}$$

and $P := P_> \oplus P_< \oplus \widetilde{P}$ acting on $H = Q_>H \oplus Q_<H \oplus \widetilde{H}$.

[3] Compare with the discussion after Lemma 3.2 in [Brü92]. Again, the dependence on C_4 being small enough has been removed.

Lemma 3.1.5. *The adjoints of the operators defined in Definition 3.1.4 are given by*

$$
\begin{aligned}
P_>^* f(x) &= \int_x^{s_0} (W_>(y,x))^* f(y) dy, && f \in L^2((0,s_0], Q_> H)\\
P_<^* f(x) &= -\int_0^x (W_<(y,x))^* f(y) dy, && f \in L^2((0,s_0], Q_< H)\\
\widetilde{P}^* f(x) &= -\bigoplus_{\substack{\lambda\in \operatorname{spec}\widetilde{S}\\ \lambda \geqslant \frac{1}{2}}} P_{0,\lambda} f(x) \oplus \bigoplus_{\substack{\lambda\in \operatorname{spec}\widetilde{S}\\ \lambda < \frac{1}{2}}} P_{1,\lambda} f(x) && f \in L^2((0,s_0], \widetilde{H}).
\end{aligned}
$$

Proof. Let $f,\ g \in L^2((0,s_0], Q_> H)$.

$$
\begin{aligned}
(P_> f, g)_{L^2(I,Q_>H)} &= \int_0^{s_0} \left\langle \int_0^x W_>(x,y) f(y) dy, g(x) \right\rangle_{Q_>H} dx\\
&\overset{\text{Fubini}}{=} \int_0^{s_0} \left\langle f(y), \int_y^{s_0} (W_>(x,y))^* g(x) dx \right\rangle_{Q_>H} dy = (f, P_>^* g)_{L^2(I,Q_>H)}.
\end{aligned}
$$

The other cases are treated analogously. □

Theorem 3.1.6. [4] *The parametrix P has the following properties:*

1. *For $f \in C_0(I, H_1 \oplus \widetilde{H})$ we have $Pf \in C^1(I, H_{tot}) \cap C(I, H_1 \oplus \widetilde{H})$ and*

$$\mathrm{T}_{tot} Pf(x) = f(x), \quad x \in I.$$

For $f \in C_0(I, H_1 \oplus \widetilde{H})$ we have $P^* f \in C^1(I, H \oplus \widetilde{H}) \cap C(I, H_1 \oplus \widetilde{H})$ and

$$\underbrace{\left(-\frac{\partial}{\partial x} + x^{-\beta} S(x)\right) \oplus \left(-\frac{\partial}{\partial x} + x^{-1}\widetilde{S}\right)}_{=T_{tot}^t} P^* f(x) = f(x), \quad x \in I.$$

2. *For $f \in L^2(I, H_{tot})$ we have $Pf,\ P^* f \in C(I, H_{tot})$. For $f \in L^2(I, H)$*

$$
\begin{aligned}
\|(P_> Q_> f)(x)\|_{Q_>H} + \|(P_>^* Q_> f)(x)\|_{Q_>H} &\leqslant C x^{\frac{\beta}{2}} \|Q_> f\|_{L^2(I,Q_>H)},\\
\|(P_< Q_< f)(x)\|_{Q_<H} + \|(P_<^* Q_< f)(x)\|_{Q_<H} &\leqslant C x^{\frac{\beta}{2}} \|Q_< f\|_{L^2(I,Q_<H)},
\end{aligned}
$$

for all $x \in I$. For $f \in L^2(I, \widetilde{H})$ and the eigenprojection $Q_{-\frac{1}{2}} : \widetilde{H} \to \ker(\widetilde{S} + \frac{1}{2})$

$$
\begin{aligned}
\left\|\widetilde{P} f(x)\right\|_{\widetilde{H}} &\leqslant C x^{\frac{1}{2}} \left\|(\mathbb{1} - Q_{-\frac{1}{2}}) f\right\|_{L^2(I,\widetilde{H})} + x^{\frac{1}{2}} |\log x|^{\frac{1}{2}} \left\|Q_{-\frac{1}{2}} f\right\|_{L^2(I,\widetilde{H})}\\
\left\|\widetilde{P}^* f(x)\right\|_{\widetilde{H}} &\leqslant C x^{C_1} \left\|(\mathbb{1} - Q_{-\frac{1}{2}}) f\right\|_{L^2(I,\widetilde{H})} + x^{\frac{1}{2}} |\log x|^{\frac{1}{2}} \left\|Q_{-\frac{1}{2}} f\right\|_{L^2(I,\widetilde{H})},
\end{aligned}
$$

where $C_1 := \min\left\{\left\{\lambda \in \operatorname{spec}\widetilde{S} \;\middle|\; \lambda > -\frac{1}{2}\right\}, \frac{1}{2}\right\}$. In particular, this implies that P and P^* are bounded operators on $L^2(I, H_{tot})$.

[4] This theorem is comparable with Theorem 3.1 in [Brü92] which is stated for the cone case. The statement and the proof have been adapted to the horn case.

3. *With $\varphi \in C_0^\infty(I)$ we have*

$$\|\varphi P f\|^2_{L^2(I,H_1\oplus\widetilde{H})} + \|\varphi P f\|^2_{H^1(I,H_{tot})} \leqslant C_\varphi \|f\|^2_{L^2(I,H_{tot})} \quad \forall f \in L^2(I,H_{tot}).$$

4. *$P : L^2(I,H_{tot}) \to L^2(I,H_{tot})$ is compact.*

Proof. 1. The cases $\widetilde{P}$ and $\widetilde{P}^*$ are trivial. Theorem 3.2 in [Kre71] (page 196) proves the assertion for $P_>$ and $P_<^*$.

$P_<$: Let $f \in C_0(I, Q_< H_1)$ and $g \in C_0^\infty(I, Q_< H_1)$.

$$\begin{aligned}
(T_< P_< f, g)_{L^2(I,Q_<H)} &= \int_0^{s_0} \left\langle \left(\frac{\partial}{\partial x} + x^{-\beta} S_<(x)\right) \int_{s_0}^x V_>^*(y,x) f(y) dy, g(x) \right\rangle dx \\
&= \int_0^{s_0} \frac{\partial}{\partial x} \left[\int_{s_0}^x \langle f(y), V_>(y,x) g(x)\rangle \, dy\right] dx \\
&\quad + \int_0^{s_0}\int_{s_0}^x \left\langle f(y), V_>(y,x)\left(-\frac{\partial}{\partial x} + x^{-\beta} S_<(x)\right) g(x)\right\rangle dydx \\
&= \int_0^{s_0} \langle f(y), V_>(y,0) g(0)\rangle \, dy \\
&\quad + \int_0^{s_0}\int_{s_0}^x \left\langle f(y), -V_>(y,x) g'(x) - \frac{\partial V_>(y,x)}{\partial x} g(x)\right\rangle dydx \\
&\overset{Fubini}{=} \int_0^y \frac{\partial}{\partial x} \langle f(y), V_>(y,x) g(x)\rangle \, dxdy \\
&= \int_0^{s_0} \left\langle f(y), \underbrace{V_>(y,y)}_{=\mathbb{1}} g(y) \right\rangle dy = (f,g)_{L^2(I,Q_<H)}
\end{aligned}$$

$P_>^*$: With $T_>^t = -\frac{\partial}{\partial x} + x^{-\beta} S_>(x)$, $f \in C_0(I, Q_> H_1)$ and $g \in C_0^\infty(I, Q_> H_1)$ we find

$$\begin{aligned}
\left(T_>^t P_>^* f, g\right)_{L^2(I,Q_>H)} &= \int_0^{s_0} \left\langle \left(-\frac{\partial}{\partial x} + x^{-\beta} S_>(x)\right) \int_x^{s_0} (W_>(y,x))^* f(y) dy, g(x) \right\rangle dx \\
&= -\int_0^{s_0} \frac{\partial}{\partial x}\left[\int_x^{s_0} \langle f(y), W_>(y,x) g(x)\rangle \, dy\right] dx \\
&\quad + \int_0^{s_0}\int_x^{s_0} \left\langle f(y), W_>(y,x) g'(x) + \underbrace{W_>(y,x) x^{-\beta} S_>(x)}_{=\frac{\partial}{\partial x} W_>(y,x)} g(x) \right\rangle dydx \\
&\overset{Fubini}{=} \int_0^{s_0} \langle f(y), W_>(y,0) g(0)\rangle \, dx + \int_0^{s_0}\int_0^y \frac{\partial}{\partial x} \langle f(y), W_>(y,x) g(x)\rangle \, dxdy \\
&= \int_0^{s_0} \langle f(y), W_>(y,y) g(y)\rangle \, dy = (f,g)_{L^2(I,Q_>H)}.
\end{aligned}$$

2. We apply the estimates proved in Lemma 3.1.2 and corollary 3.1.3.

- Let $f \in L^2(I, Q_> H)$:

$$\begin{aligned}
\|(P_> f)(x)\|_{Q_>H} &\leqslant \int_0^x \|W_>(x,y)\|_{\mathcal{L}(Q_>H)} \|f(y)\|_{Q_>H}\, dy \\
&\overset{3.1.2,\ CS}{\leqslant} C \left(\int_0^x e^{-2dF(x,y)} \left(F(x,y)^{-\varepsilon} + 1\right)^2 dy \right)^{\frac{1}{2}} \|f\|_{L^2(I,Q_>H)} \\
&\leqslant C \left(x^\beta \int_0^x y^{-\beta} e^{-2dF(x,y)} \left(F(x,y)^{-\varepsilon} + 1\right)^2 dy \right)^{\frac{1}{2}} \|f\|_{L^2(I,Q_>H)} \\
&= C x^{\frac{\beta}{2}} \left(\int_0^\infty e^{-2ds} \left(s^{-\varepsilon} + 1\right)^2 ds \right)^{\frac{1}{2}} \|f\|_{L^2(I,Q_>H)} \overset{-2\varepsilon > -1}{=} C x^{\frac{\beta}{2}} \|f\|_{L^2(I,Q_>H)} .
\end{aligned}$$

- We proceed with the estimate for $P_<$: Let $f \in L^2(I, Q_< H)$.

$$\begin{aligned}
\|(P_< f)(x)\|_{Q_<H} &\leqslant \int_x^{s_0} \|W_<(x,y)\|_{\mathcal{L}(Q_<H)} \|f(y)\|_{Q_<H}\, dy \\
&\overset{3.1.3,\ CS}{\leqslant} C \left(\int_x^{s_0} e^{-2dF(y,x)} \left(F(y,x)^{-\varepsilon} + 1\right)^2 dy \right)^{\frac{1}{2}} \|f\|_{L^2(I,Q_<H)}
\end{aligned}$$

We get an estimate for the integral by splitting it into two parts. [5]

$$\begin{aligned}
\int_x^{2x} e^{-2dF(y,x)} \left(F(y,x)^{-\varepsilon} + 1\right)^2 dy &\leqslant (2x)^\beta \int_0^\infty e^{-2ds} \left(s^{-\varepsilon} + 1\right)^2 ds \leqslant C x^\beta \\
\int_{2x}^{s_0} e^{-2dF(y,x)} \left(F(y,x)^{-\varepsilon} + 1\right)^2 dy &\leqslant s_0^\beta \int_{F(2x,x)}^{F(s_0,x)} e^{-2ds} \left(s^{-\varepsilon} + 1\right)^2 ds \\
&\leqslant s_0^\beta \left(F(2x,x)^{-\varepsilon} + 1\right)^2 \int_{F(2x,x)}^{F(s_0,x)} e^{-2ds} ds \\
&= s_0^\beta \left[\left(\frac{1 - 2^{-\beta+1}}{\beta - 1} x^{-\beta+1} \right)^{-\varepsilon} + 1 \right]^2 \frac{1}{2d} (e^{-2dF(2x,x)} - e^{-2dF(s_0,x)}) \\
&\leqslant K_1 e^{-K_2 x^{-\beta+1}} \leqslant C x^\beta
\end{aligned}$$

The last estimate uses l'Hôpital's rule:

$$\begin{aligned}
\lim_{x\to 0} \frac{x^{-\beta}}{e^{K_2 x^{-\beta+1}}} &= \lim_{x\to 0} \frac{-\beta x^{-1}}{K_2(-\beta+1) e^{K_2 x^{-\beta+1}}} = \lim_{x\to 0} \frac{\beta x^{\beta-2}}{K_2^2(-\beta+1)^2 e^{K_2 x^{-\beta+1}}} = \cdots \\
&= K_n \lim_{x\to 0} \frac{x^{(n-1)\beta - n}}{e^{K_2 x^{-\beta+1}}} = 0 \qquad \text{for } n \in \mathbb{N} \text{ big enough, since } \beta > 1.
\end{aligned}$$

Thus, we have proved:

$$\|(P_< f)(x)\|_{Q_<H} \leqslant C x^{\frac{\beta}{2}} \|f\|_{L^2(I,Q_<H)} .$$

[5] The idea of splitting the integral is taken from the proof of Lemma 3.1 (ii) in [LP98].

- At last, for $f \in L^2(I, \widetilde{H})$ Lemma 2.1 in [BS88] implies

$$\left\| \left(\widetilde{P} \left(\mathbb{1} - Q_{-\frac{1}{2}} \right) f \right)(x) \right\|_{\widetilde{H}} \leqslant C x^{\frac{1}{2}} \left\| \left(\mathbb{1} - Q_{-\frac{1}{2}} \right) f \right\|_{L^2(I,\widetilde{H})} \qquad \text{and}$$
$$\left\| \left(\widetilde{P} Q_{-\frac{1}{2}} f \right)(x) \right\|_{\widetilde{H}} \leqslant x^{\frac{1}{2}} |\log x|^{\frac{1}{2}} \left\| Q_{-\frac{1}{2}} f \right\|_{L^2(I,\widetilde{H})}.$$

The estimates for the adjoint operators are found analogously with the following exception. Lemma 2.1 in [BS88] yields

$$\left\| \left(\widetilde{P}^* \left(\mathbb{1} - Q_{-\frac{1}{2}} \right) f \right)(x) \right\|_{\widetilde{H}} \leqslant C x^{C_1} \left\| \left(\mathbb{1} - Q_{-\frac{1}{2}} \right) f \right\|_{L^2(I,\widetilde{H})} \qquad \text{and}$$
$$\left\| \left(\widetilde{P}^* Q_{-\frac{1}{2}} f \right)(x) \right\|_{\widetilde{H}} \leqslant x^{\frac{1}{2}} |\log x|^{\frac{1}{2}} \left\| Q_{-\frac{1}{2}} f \right\|_{L^2(I,\widetilde{H})},$$

where $C_1 := \min \left\{ \left\{ \lambda \in \operatorname{spec} \widetilde{S} \;\middle|\; \lambda > -\frac{1}{2}, \right\}, \frac{1}{2} \right\}$.

3. We prove the estimate separately for $P_>$, $P_<$ and $\widetilde{P}$. Let $u := \varphi P_> f$ with $f \in C_0^\infty(I, Q_> H)$ and $\varphi \in C_0^\infty(I)$ with $\operatorname{supp} \varphi \subset [a, s_0]$, $0 < a < s_0$. We begin with some auxiliary calculations:

$$\begin{aligned}
\operatorname{Re}(T_> u, u)_{L^2(I,Q_>H)} &= \frac{1}{2} \left((T_> u, u)_{L^2(I,Q_>H)} + (u, T_> u)_{L^2(I,Q_>H)} \right) \\
&= \frac{1}{2} \int_0^{s_0} \langle u'(x) + x^{-\beta} S_>(x) u(x), u(x) \rangle dx + \frac{1}{2} \int_0^{s_0} \langle u(x), u'(x) + x^{-\beta} S_>(x) u(x) \rangle dx \\
&= \underbrace{\frac{1}{2} \int_0^{s_0} \frac{\partial}{\partial x} \langle u(x), u(x) \rangle dx}_{=0} + \int_0^{s_0} x^{-\beta} \langle S_>(x) u(x), u(x) \rangle dx \geqslant s_0^{-\beta} \left\| (S_>)^{\frac{1}{2}} u \right\|^2_{L^2(I,Q_>H)}
\end{aligned}$$

$$\begin{aligned}
\left| 2 \operatorname{Re}(u', x^{-\beta} S_> u)_{L^2(I,Q_>H)} \right| &= \left| (u', x^{-\beta} S_> u) + (x^{-\beta} S_> u, u') \right| = \left| -(u, (x^{-\beta} S_>)' u) \right| \\
&= \left| \int_0^{s_0} \beta x^{-\beta-1} \langle S_>(x) u(x), u(x) \rangle dx - \int_0^{s_0} x^{-\beta} \underbrace{\langle S'_>(x) u(x), u(x) \rangle}_{= \langle S'_>(x) S_>(x)^{-1} S_>(x) u(x), u(x) \rangle} dx \right| \\
&\leqslant \beta a^{-\beta-1} \left\| (S_>)^{\frac{1}{2}} u \right\|^2_{L^2(I,Q_>H)} + \int_0^{s_0} \underbrace{x^{-\beta} \alpha(x)}_{\leqslant \sup_{x \in [a,s_0]} x^{-\beta} \alpha(x)} \left\| (S_>(x))^{\frac{1}{2}} u(x) \right\|^2 dx \\
&\leqslant C_\varphi \left\| (S_>)^{\frac{1}{2}} u \right\|^2_{L^2(I,Q_>H)}.
\end{aligned}$$

Combining the two results, gives

$$\begin{aligned}
\left| 2 \operatorname{Re}(u', x^{-\beta} S_> u)_{L^2(I,Q_>H)} \right| &\leqslant C_\varphi \left\| (S_>)^{\frac{1}{2}} u \right\|^2_{L^2(I,Q_>H)} \leqslant C_\varphi s_0^\beta \operatorname{Re}(T_> u, u)_{L^2(I,Q_>H)} \\
&\leqslant C_\varphi s_0^\beta \|T_> u\|_{L^2(I,Q_>H)} \|u\|_{L^2(I,Q_>H)} \leqslant \frac{C_\varphi s_0^\beta}{2} \left(\|T_> u\|^2_{L^2(I,Q_>H)} + \|u\|^2_{L^2(I,Q_>H)} \right)
\end{aligned}$$

and

$$\|T_{>}u\|^2_{L^2(I,Q_>H)} = \left\|u'\right\|^2_{L^2(I,Q_>H)} + \left\|x^{-\beta}S_{>}u\right\|^2_{L^2(I,Q_>H)} + 2\operatorname{Re}(u', x^{-\beta}S_{>}u)_{L^2(I,Q_>H)}$$
$$\geqslant \left\|u'\right\|^2_{L^2(I,Q_>H)} + C_2^2 s_0^{-2\beta}\left\|u\right\|^2_{L^2(I,Q_>H)} - \frac{C_\varphi s_0^\beta}{2}\left(\|T_{>}u\|^2_{L^2(I,Q_>H)} + \|u\|^2_{L^2(I,Q_>H)}\right).$$

Finally, this implies

$$\left\|u'\right\|^2_{L^2(I,Q_>H)} \leqslant \left(1+\frac{C_\varphi s_0^\beta}{2}\right)\|T_{>}u\|^2 + \left(\frac{C_\varphi s_0^\beta}{2} - C_2^2 s_0^{-2\beta}\right)\|u\|^2 \leqslant \widetilde{C}_\varphi\left(\|T_{>}u\|^2 + \|u\|^2\right).$$

Inserting $u = \varphi P_{>}f$, gives:

$$\begin{aligned}
\left\|\frac{\partial}{\partial x}(\varphi P_{>}f)\right\|_{L^2(I,Q_>H)} &\leqslant \widetilde{C}_\varphi\left(\|T_{>}\varphi P_{>}f\|_{L^2(I,Q_>H)} + \|\varphi P_{>}f\|_{L^2(I,Q_>H)}\right)\\
&\overset{1.}{\leqslant} \widetilde{C}_\varphi\left(\left\|\varphi' P_{>}f\right\|_{L^2(I,Q_>H)} + \|\varphi f\|_{L^2(I,Q_>H)} + \|\varphi P_{>}f\|_{L^2(I,Q_>H)}\right)\\
&\overset{2.}{\leqslant} \widetilde{C}_\varphi\left(\sup_{x\in I}|\varphi'(x)|\,\|P_{>}\| + \sup_{x\in I}|\varphi(x)|(1+\|P_{>}\|)\right)\|f\|_{L^2(I,Q_>H)}\\
&=: \widetilde{\widetilde{C}}_\varphi\,\|f\|_{L^2(I,Q_>H)}\\
\Rightarrow\quad \|\varphi P_{>}f\|_{H^1(I,Q_>H)} &\leqslant \left(\widetilde{\widetilde{C}}_\varphi + \|P_{>}\|_{\mathcal{L}(L^2(I,Q_>H))}\right)\|f\|_{L^2(I,Q_>H)}.
\end{aligned}$$

If we replace $>$ by $<$ in the computations above, we get the analogous result for $\varphi P_{<}f$. The estimate for $\widetilde{P}$ follows because $\widetilde{S}$ is a symmetric matrix.

We proceed with the second assertion.

$$\begin{aligned}
S(x)(\varphi Pf)(x) &= \varphi(x)S(x)(Pf)(x) = \varphi(x)x^\beta T(Pf)(x) - \varphi(x)x^\beta\frac{\partial}{\partial x}(Pf)(x)\\
&\overset{1.}{=} x^\beta\varphi(x)f(x) - x^\beta\frac{\partial}{\partial x}(\varphi Pf)(x) + x^\beta\varphi'(x)(Pf)(x)\\
\Rightarrow\qquad \|(\varphi Pf)(x)\|^2_{H_1} &= \|S(x)(\varphi Pf)(x)\|^2_H + \|(\varphi Pf)(x)\|^2_H\\
&\leqslant C'_\varphi\left(\left\|\frac{\partial}{\partial x}(\varphi Pf)(x)\right\|^2_H + \|f(x)\|^2_H\right)\\
\Rightarrow\qquad \|\varphi Pf\|^2_{L^2(I,H_1)} &\leqslant C'_\varphi\left(\left\|\frac{\partial}{\partial x}(\varphi Pf)\right\|^2_{L^2(I,H)} + \|f\|^2_{L^2(I,H)}\right) \leqslant \widetilde{\widetilde{C'}}_\varphi\,\|f\|^2_{L^2(I,H)}
\end{aligned}$$

4. Let $\psi \in C_0^\infty(I,[0,1])$ with

$$\psi(x) = \begin{cases} 0 & x\in(0,a]\\ 1 & x\in[b,s_0]\end{cases} \qquad\text{with}\qquad 0<a<b<s_0.$$

From the third part of this proof follows that

$$\psi P : L^2(I,H_{\text{tot}}) \overset{\text{continuous}}{\to} H^1(I,H_{\text{tot}}) \overset{\text{compact}}{\hookrightarrow} L^2(I,H_{\text{tot}})$$

and thus, $\psi P : L^2(I, H_{\mathrm{tot}}) \to L^2(I, H_{\mathrm{tot}})$ is compact. Furthermore, the second part of the proof shows that

$$\begin{aligned}\|(1-\psi)Pf\|_{L^2(I,H_{\mathrm{tot}})} &\leqslant C\left(\int_0^{s_0}(1-\psi(x))^2 x|\log x| dx\right)^{\frac{1}{2}} \|f\|_{L^2(I,H_{\mathrm{tot}})} \\ &\leqslant \frac{C}{\sqrt{2}} \sup_{z\in[0,b]} z|\log z|^{\frac{1}{2}} \|f\|_{L^2(I,H_{\mathrm{tot}})}.\end{aligned}$$

We define ψ_n with $a := \frac{1}{n+1}$ and $b := \frac{1}{n}$ and find $\psi_n P \overset{n\to\infty}{\to} P$. Since compact operators are closed inside the space of bounded operators, the assertion is proved. $\square$

3.2 Fredholm property

In this section unitary functions are constructed that transform any operator with horn singularity D into a "reduced" operator with horn singularity D_c, where

$$D_c|_U \cong \left(\frac{\partial}{\partial x} + x^{-\beta} S_c(x)\right) \oplus \left(\frac{\partial}{\partial x} + x^{-1}\widetilde{S}\right), \quad Q_>(x) \equiv Q_>(s_0), \text{ and } Q_<(x) \equiv Q_<(s_0).$$

Then the parametrix defined in Section 3.1 and the generic parametrix on M_1 are added to define a parametrix for $D_{\min}$. This proves that $D_{\min}$ is a Fredholm operator and

$$\operatorname{ind} D_{\min} = \operatorname{ind}(D_c)_{\min}.$$

Furthermore, the closed extensions of $D_{\min}$ are discussed.

Lemma 3.2.1. [6] *Let $P \in C^1((0, s_0], \mathcal{L}(H))$ be a family of orthogonal projections. Then there is an operator-valued function $U \in C^1((0, s_0], \mathcal{L}(H))$ which satisfies:*

1. *$U(x)^{-1}$ exists and is continuously differentiable for all $x \in I$.*
2. *$U(x)P(s_0)U(x)^{-1} = P(x)$ for all $x \in I$.*
3. *$U'(x) = [P'(x), P(x)]\, U(x)$ for all $x \in I$.*
4. *$U(x)$ is unitary for all $x \in I$.*
5. *$\|U'(x)P(s_0)\| \leqslant \|P'(x)\|$ and $\|U'(x)(\mathbb{1} - P(s_0))\| \leqslant \|P'(x)\|$ for all $x \in I$.*

Proof. Differentiating $P^2 = P$, gives

$$\begin{aligned} & & P'P + PP' &= P' & &| \cdot P \\ &\Rightarrow & P'P + PP'P &= P'P & &| - P'P \\ &\Rightarrow & PP'P &= 0. & & \end{aligned} \tag{3.10}$$

[6]This lemma and its proof are taken from [Kat95] II §4 2 page 99ff. We adapted the statement and the proof to our case.

We define $Q := [P', P] = P'P - PP'$ and calculate

$$\begin{aligned} QP =& P'P^2 - PP'P \overset{(3.10)}{=} P'P \\ PQ =& PP'P - P^2P' \overset{(3.10)}{=} -PP' \\ \Rightarrow \quad [Q,P] =& P'P + PP' = (P^2)' = P'. \end{aligned}$$

The linear differential equations

$$U' = QU, \quad U(s_0) = \mathbb{1} \qquad \text{and} \tag{3.11}$$
$$V' = -VQ, \quad V(s_0) = \mathbb{1} \tag{3.12}$$

have unique continuously differentiable solutions also denoted by U and V, respectively. On the one hand,

$$(VU)' = V'U + VU' = -VQU + VQU = 0 \qquad \Rightarrow \qquad V(x)U(x) = V(s_0)U(s_0) = \mathbb{1}.$$

On the other hand,

$$(UV)' = U'V + UV' = QUV - UVQ = [Q, UV] \quad \text{and} \quad U(s_0)V(s_0) = \mathbb{1}.$$

Since the constant function $Z(x) = \mathbb{1}$ satisfies the linear differential equation $Z' = [Q, Z]$ with initial condition $Z(s_0) \equiv \mathbb{1}$ and since that solution is unique, we find $U(x)V(x) = Z(x) = \mathbb{1}$. Taking both relations together, we find that $U(x)^{-1} = V(x)$.

PU solves the differential equation (3.11) with a different initial value

$$(PU)' = P'U + PU' = [Q,P]U + PQU = QPU - PQU + PQU = QPU \text{ and}$$
$$P(s_0)U(s_0) = P(s_0).$$

Since this equation is also solved by $U(x)P(s_0)$, from the uniqueness of the solution follows $P(x)U(x) = U(x)P(s_0)$ which gives the second part of the claim.
Since P is an orthogonal projection, we find

$$Q^* = \big(P'P - PP'\big)^* \overset{P=P^*}{=} PP' - P'P = -(P'P - PP') = -Q.$$

Thus, U^* solves the differential equation

$$(U^*)' \overset{(3.11)}{=} (QU)^* = U^*Q^* = -U^*Q, \quad U^*(s_0) = \mathbb{1}^* = \mathbb{1}$$

which is equal to the differential equation (3.12) that is solved by U^{-1}. Thus, uniqueness proves that $U^* = U^{-1}$.
Finally, the norm estimates for the derivative are computed:

$$\begin{aligned} \big\|U'(x)P(s_0)\big\| &\overset{(3.11)}{=} \|Q(x)U(x)P(s_0)\| = \big\|\big[P'(x)P(x) - P(x)P'(x)\big]\, P(x)U(x)\big\| \\ &\overset{(3.10)}{=} \big\|P'(x)P(x)U(x)\big\| \leqslant \big\|P'(x)\big\| \end{aligned}$$

$$\begin{aligned}\left\|U'(x)(\mathbb{1}-P(s_0))\right\| &\overset{(3.11)}{=} \left\|\left[P'(x)P(x)-P(x)P'(x)\right](\mathbb{1}-P(x))U(x)\right\| \\ &\overset{(3.10)}{=} \left\|-P(x)P'(x)U(x)\right\| \leqslant \left\|P'(x)\right\|\end{aligned}$$

□

Lemma 3.2.2. [7] *There is a smooth family $U(x)$ of unitary operators in H, such that*

$$S_c(x) := U^*(x)S(x)U(x)$$

satisfies

$$\begin{aligned}&S_c(x)Q_>(s_0) \geqslant C_2 Q_>(s_0) \quad \textit{and}\\ &S_c(x)Q_<(s_0) \leqslant -C_2 Q_<(s_0).\end{aligned}$$

Furthermore,

$$\begin{aligned}\left\|U'(x)Q_>(s_0)\right\|_{\mathcal{L}(H)} \leqslant \left\|Q'_>(x)\right\|_{\mathcal{L}(H)}, &\qquad \left\|U'(x)Q_<(s_0)\right\|_{\mathcal{L}(H)} \leqslant \left\|Q'_<(x)\right\|_{\mathcal{L}(H)}\\ \left\|(S_c)'_>(x)(S_c)_>(x)^{-1}\right\|_{\mathcal{L}(H)} \leqslant \alpha(x) \quad \textit{and} &\qquad \left\|(S_c)'_<(x)(S_c)_<(x)^{-1}\right\|_{\mathcal{L}(H)} \leqslant \alpha(x).\end{aligned}$$

Proof. We apply Lemma 3.2.1 to the family of projections $Q_>(x)$ which gives us the unitary family $U(x)$ with

$$U^*(x)Q_>(x)U(x) = Q_>(s_0) \tag{3.13}$$

and

$$U^*(x)Q_<(x)U(x) = U^*(x)(\mathbb{1}-Q_>(x))U(x) = \mathbb{1}-Q_>(s_0) = Q_<(s_0).$$

For every $v \in H$ we find

$$\begin{aligned}\langle[S_c(x)-C_2]Q_>(s_0)v,v\rangle &= \langle[U^*(x)S(x)U(x)-C_2\underbrace{U^*(x)U(x)}_{=\mathbb{1}}]Q_>(s_0)v,v\rangle\\ &= \langle[S(x)-C_2]U(x)Q_>(s_0)v,U(x)v\rangle\\ &\overset{(3.13)}{=} \langle[S(x)-C_2]Q_>(x)U(x)v,U(x)v\rangle \geqslant 0,\end{aligned}$$

i.e. $S_c(x)Q_>(s_0) \geqslant C_2Q_>(s_0)$. Analogously, $S_c(x)Q_<(s_0) \leqslant -C_2Q_<(s_0)$ is proved. The estimates for $U'(x)Q_>(s_0)$ and $U'(x)Q_<(s_0) = U'(x)(\mathbb{1}-Q_>(s_0))$ follow directly from Lemma

[7] This lemma is partially equal to Lemma 3.5 on page 280 in [Brü92]. We adapted the statement to the horn operator case and added further details to the proof.

3.2.1, 5. At last, we compute $(S_c)'_>$:

$$\begin{aligned}
(S_c)'_>(x) &= (Q_>(s_0)S_c(x))' = (Q_>(s_0)U^*(x)S(x)U(x)Q_>(s_0))' \\
&= -Q_>(s_0)U^*(x)U'(x)U^*(x)S(x)U(x)Q_>(s_0) + Q_>(s_0)U^*(x)S'(x)U(x)Q_>(s_0) \\
&\quad + Q_>(s_0)U^*(x)S(x)U'(x)Q_>(s_0) \\
&\overset{3.2.1,\ 3.}{=} -U^*(x)\underbrace{Q_>(x)\left[Q'_>(x), Q_>(x)\right]Q_>(x)}_{=0}U(x)U^*(x)S(x)U(x) \\
&\quad + Q_>(s_0)U^*(x)S'(x)U(x)Q_>(s_0) \\
&\quad + U^*(x)S(x)\underbrace{Q_>(x)\left[Q'_>(x), Q_>(x)\right]Q_>(x)}_{=0}U(x) \\
&= Q_>(s_0)U^*(x)S'(x)U(x)Q_>(s_0).
\end{aligned}$$

This implies

$$\begin{aligned}
\left\|(S_c)'_>(x)(S_c)_>(x)^{-1}\right\| &= \left\|Q_>(s_0)U^*(x)S'(x)\underbrace{U(x)U^*(x)}_{=\mathbb{1}}S(x)^{-1}U(x)\right\| \\
&\leqslant \left\|S'(x)S(x)^{-1}\right\| = \alpha(x).
\end{aligned}$$

Replacing $>$ by $<$, gives the analogous result for $(S_c)_<$. $\square$

Lemma 3.2.3. [8] *Let D be an operator with horn singularity. Define*

$$\mathcal{D}_1 := \left\{ \begin{array}{l} \widehat{u} \in \mathcal{D}(D_{\max}) \\ \widehat{u}|_U = u + \widetilde{u} \\ \quad \in L^2(I, H \oplus \widetilde{H}) \end{array} \;\middle|\; \begin{array}{r} \|u(x)\|_H = O\left(x^{\frac{\beta}{2}}\right) \\ \left\|\left(Q - Q_{-\frac{1}{2}}\right)\widetilde{u}(x)\right\|_{\widetilde{H}} = O\left(x^{\frac{1}{2}}\right) \text{ and} \\ \left\|Q_{-\frac{1}{2}}\widetilde{u}(x)\right\|_{\widetilde{H}} = O\left(x^{\frac{1}{2}}|\log x|^{\frac{1}{2}}\right) \text{ for } x \to 0 \end{array} \right\}.$$

Let D_c be the operator

$$D_c|_U \cong T_c \oplus \widetilde{T}_c := \left(\frac{\partial}{\partial x} + x^{-\beta}S_c(x)\right) \oplus \left(\frac{\partial}{\partial x} + x^{-1}\widetilde{S}\right)$$

and

$$\mathcal{D}_1^c := \left\{ \begin{array}{l} \widehat{u} \in \mathcal{D}((D_c)_{\max}) \\ \widehat{u}|_U = u + \widetilde{u} \\ \quad \in L^2(I, H \oplus \widetilde{H}) \end{array} \;\middle|\; \begin{array}{r} \|u(x)\|_H = O\left(x^{\frac{\beta}{2}}\right) \\ \left\|\left(Q - Q_{-\frac{1}{2}}\right)\widetilde{u}(x)\right\|_{\widetilde{H}} = O\left(x^{\frac{1}{2}}\right) \text{ and} \\ \left\|Q_{-\frac{1}{2}}\widetilde{u}(x)\right\|_{\widetilde{H}} = O\left(x^{\frac{1}{2}}|\log x|^{\frac{1}{2}}\right) \text{ for } x \to 0 \end{array} \right\}.$$

[8] The content of this lemma is similar to Lemma 3.3 in [Brü92], but we adapted it to the horn case and extended it to the non-constant case. The proof of the non-constant case is similar to part of the proof of Theorem 3.2 in [Brü92].

Then $\mathcal{D}_1 = U\mathcal{D}_1^c$, where U is the unitary transformation function from Lemma 3.2.2 extended by identity. For $u \in \mathcal{D}_1$ we have

$$\begin{aligned} &\forall \varphi \in C_0^\infty([0,s_0)) \qquad \varphi U^* u = P(D_c)_{\max} \varphi U^* u \\ \exists\, s \in I,\ &\forall \varphi \in C_0^\infty([0,s)) \qquad \varphi u = \widehat{\mathrm{P}}\, V D_{\max} \varphi u \end{aligned} \tag{3.14}$$

where $\widehat{\mathrm{P}} = UPU^*$ and $V \in \mathcal{L}(L^2((0,s),H))$.

Proof.

$$\begin{aligned} U^*(x)TU(x) &= U^*(x)U'(x) + \underbrace{U^*(x)U(x)}_{=\mathbb{1}} \frac{\partial}{\partial x} + x^{-\beta} \underbrace{U^*(x)S(x)U(x)}_{=S_c(x)} + U^*(x)S_1(x)U(x) \\ &= \left(\frac{\partial}{\partial x} + x^{-\beta} S_c(x)\right) + U^*(x)\left[U'(x)U^*(x) + S_1(x)\right] U(x) \\ &=: T_c + U^*(x)\left[B(x) + S_1(x)\right] U(x) \\ \widetilde{T} &= \frac{\partial}{\partial x} + x^{-1}\widetilde{S} + \widetilde{S}_1(x) = T_c + \widetilde{S}_1(x) \end{aligned}$$

We extend $B \oplus \mathbb{1}$ to an operator $B : L^2(M,E) \to L^2(M,F)$ by setting it equal to zero on the rest of the manifold. Let S_1 denote the operator that is equal to the operator family $I \ni x \mapsto S_1(x) \oplus \widetilde{S}_1$ on U. Thus,

$$D = UD_cU^* + B + S_1. \tag{3.15}$$

Next, the equality $\mathcal{D}_1 = U\mathcal{D}_1^c$ is proved. It suffices to show that every $\widehat{u} \in \mathcal{D}_1$ with $\widehat{u}|_U = u + \widetilde{u}$ satisfies $(B + S_1)u \in L^2(I,H)$ and $\widetilde{S}_1\widetilde{u} \in L^2(I,\widetilde{H})$. Applying Lemma 3.2.2, we find

$$\begin{aligned} \int_0^{s_0} \|B(x)u(x)\|_H^2\, dx &= \int_0^{s_0} \left\|U'(x)U^*(x)u(x)\right\|_H^2 dx \\ &= \int_0^{s_0} \left\|U'(x)(Q_>(s_0) + Q_<(s_0))U^*(x)u(x)\right\|_H^2 dx \\ &\overset{3.2.2}{\leqslant} 2\int_0^{s_0} \left\|Q'_>(x)\right\|^2 \|Q_>(x)u(x)\|^2 + \left\|Q'_<(x)\right\|^2 \|Q_<(x)u(x)\|^2\, dx \\ &\overset{\widehat{u}\in\mathcal{D}_1}{\leqslant} C\int_0^{s_0} \left(\left\|Q'_>(x)\right\|^2 + \left\|Q'_<(x)\right\|^2\right) x^\beta dx \overset{2.1.1,\ 3.}{<} \infty \end{aligned}$$

and by definition of S_1 follows

$$\begin{aligned} \int_0^{s_0} \|S_1(x)u(x)\|_H^2\, dx &\leqslant \int_0^{s_0} \|S_1(x)\|_{\mathcal{L}(H)}^2 \|u(x)\|_H^2\, dx \\ &\overset{\widehat{u}\in\mathcal{D}_1}{\leqslant} C\int_0^{s_0} \|S_1(x)\|_{\mathcal{L}(H)}^2 x^\beta dx \overset{2.1.1,\ 5.}{<} \infty \end{aligned}$$

and

$$\begin{aligned}\int_0^{s_0}\left\|\widetilde{S}_1(x)\widetilde{u}(x)\right\|_{\widetilde{H}}^2 dx &\leqslant 2\int_0^{s_0}\left\|\widetilde{S}_1(x)\left(\mathbb{1}-Q_{-\frac{1}{2}}\right)\right\|_{\mathcal{L}(\widetilde{H})}^2\left\|\left(\mathbb{1}-Q_{-\frac{1}{2}}\right)\widetilde{u}(x)\right\|_{\widetilde{H}}^2\\ &\qquad + \left\|\widetilde{S}_1(x)Q_{-\frac{1}{2}}\right\|_{\mathcal{L}(\widetilde{H})}^2\left\|Q_{-\frac{1}{2}}\widetilde{u}(x)\right\|_{\widetilde{H}}^2 dx\\ &\overset{\widehat{u}\in\mathcal{D}_1}{\leqslant} C\int_0^{s_0}\left\|\widetilde{S}_1(x)\left(\mathbb{1}-Q_{-\frac{1}{2}}\right)\right\|_{\mathcal{L}(\widetilde{H})}^2 x + \left\|\widetilde{S}_1(x)Q_{-\frac{1}{2}}\right\|_{\mathcal{L}(\widetilde{H})}^2 x|\log x| dx \overset{2.1.1,\ 7.}{<} \infty.\end{aligned}$$

We proceed by proving that $u \in \mathcal{D}_1^c$ satisfies for $\varphi \in C_0^\infty(I)$

$$\varphi u = P(D_c)_{\max}\varphi u.$$

Let $f \in C_0^\infty([0,s_0), H_1 \oplus \widetilde{H})$

$$\begin{aligned}(P(D_c)_{\max}\varphi u, f) &= ((D_c)_{\max}\varphi u, P^* f)\\ &= \int_0^{s_0}\frac{\partial}{\partial x}\langle\varphi(x)u(x),(P^*f)(x)\rangle_{H_{\mathrm{tot}}}dx + (\varphi u,(T_c^t)_{\min}P^*f)\\ &\overset{\varphi(s_0)=0}{=} -\lim_{\delta\to 0}\langle\varphi(\delta)u(\delta),(P^*f)(\delta)\rangle_{H_{\mathrm{tot}}} + (\varphi u, f) = (\varphi u, f) \quad \text{since}\\ |\langle\varphi(\delta)u(\delta),P^*f(\delta)\rangle_{H_{\mathrm{tot}}}| &\overset{3.1.6,\ 2.}{\leqslant} C\left\|\varphi\right\|_\infty \delta^{\frac{1}{2}+C_1}|\log\delta|\left\|f\right\|_{L^2(I,H)} \overset{\delta\to 0}{\longrightarrow} 0.\end{aligned}$$

This implies

$$\begin{aligned}\varphi u &= UU^*\varphi u = UP(D_c)_{\min}U^*\varphi u = UPU^*\left(D_{\min} - [B+S_1]\right)\varphi u\\ &= \widehat{\mathrm{P}}\left(D_{\min}-[B+S_1]\right)\varphi u\end{aligned}$$

and iteration gives

$$\begin{aligned}\varphi u &= \widehat{\mathrm{P}}\,D_{\min}\varphi u - \widehat{\mathrm{P}}\,[B+S_1]\,\varphi u\\ &= \widehat{\mathrm{P}}\,D_{\min}\varphi u - \widehat{\mathrm{P}}\,[B+S_1]\,\widehat{\mathrm{P}}\,D_{\min}\varphi u + \widehat{\mathrm{P}}\left([B+S_1]\,\widehat{\mathrm{P}}\right)[B+S_1]\,\varphi u\\ &= \widehat{\mathrm{P}}\left(\sum_{j=0}^{N}(-1)^j\left([B+S_1]\,\widehat{\mathrm{P}}\right)^j\right)D_{\min}\varphi u + (-1)^{N+1}\,\widehat{\mathrm{P}}\left([B+S_1]\,\widehat{\mathrm{P}}\right)^N[B+S_1]\,\varphi u.\end{aligned}$$

Let $s \in I$ and $g \in L^2((0,s],H)$:

$$\begin{aligned}\left\|B\,\widehat{\mathrm{P}}\,g\right\|_{L^2([0,s],H)}^2 &= \int_0^s\left\|U'(x)U(x)\Big(\Big[\underbrace{P_>Q_>(s_0)}_{\in Q_>(s_0)H}+\underbrace{P_<Q_<(s_0)}_{\in Q_<(s_0)H}\Big]U^*g\Big)(x)\right\|_H^2 dx\\ &\leqslant 2\int_0^s\Big[\left\|Q'_>(x)\right\|_{\mathcal{L}(H)}^2\left\|(P_>Q_>(s_0)U^*g)(x)\right\|_H^2\\ &\qquad + \left\|Q'_<(x)\right\|_{\mathcal{L}(H)}^2\left\|(P_<Q_<(s_0)U^*g)(x)\right\|_H^2\Big]dx\end{aligned}$$

$$\begin{aligned}
&\overset{3.1.6,\ 2.}{\leqslant} C \int_0^s \left(\|Q'_>(x)\|^2_{\mathcal{L}(H)} + \|Q'_<(x)\|^2_{\mathcal{L}(H)} \right) x^\beta dx \, \|g\|^2_{L^2(I,H)} \\
&=: K(s) \, \|g\|^2_{L^2(I,H)} .
\end{aligned}$$

Definition 2.1.1, 3. implies $K(s) \overset{s\to 0}{\to} 0$. Let $s \in I$ and $g \in L^2((0,s],H)$:

$$\begin{aligned}
\left\| S_1 \,\widehat{\mathrm{P}}\, g \right\|^2_{L^2([0,s],H)} &= \int_0^s \|S_1(x)U(x)\left([P_> Q_>(s_0) + P_< Q_<(s_0)]\, U^* g\right)(x)\|^2_H \, dx \\
&\leqslant 2 \int_0^s \Big[\|S_1(x)Q_>(x)\|^2_{\mathcal{L}(H)} \, \|(P_> Q_>(s_0) U^* g)(x)\|^2_H \\
&\qquad + \|S_1(x)Q_<(x)\|^2_{\mathcal{L}(H)} \, \|(P_< Q_<(s_0) U^* g)(x)\|^2_H \Big] dx \\
&\overset{3.1.6,\ 2.}{\leqslant} C \int_0^s \left(\|S_1(x)Q_>(x)\|^2_{\mathcal{L}(H)} + \|S_1(x)Q_<(x)\|^2_{\mathcal{L}(H)} \right) x^\beta dx \, \|g\|^2_{L^2(I,H)} \\
&=: K_1(s) \, \|g\|^2_{L^2(I,H)} ,
\end{aligned}$$

where $K_1(s) \overset{s\to 0}{\to} 0$ by Definition 2.1.1, 5. Let $s \in I$ and $g \in L^2((0,s],\widetilde{H})$:

$$\begin{aligned}
\left\| S_1 \,\widehat{\mathrm{P}}\, g \right\|^2_{L^2([0,s],\widetilde{H})} &= \int_0^s \left\| \widetilde{S}_1(x) \left[\left(\mathbb{1} - Q_{-\frac{1}{2}} \right) + Q_{-\frac{1}{2}} \right] \widetilde{P} g(x) \right\|^2_{\widetilde{H}} dx \\
&\leqslant 2 \int_0^s \Big[\left\| \widetilde{S}_1(x) \left(\mathbb{1} - Q_{-\frac{1}{2}} \right) \right\|^2_{\mathcal{L}(\widetilde{H})} \left\| \left(\mathbb{1} - Q_{-\frac{1}{2}} \right) \widetilde{P} g(x) \right\|^2_{\widetilde{H}} \\
&\qquad + \left\| \widetilde{S}_1(x) Q_{-\frac{1}{2}} \right\|^2_{\mathcal{L}(\widetilde{H})} \left\| Q_{-\frac{1}{2}} \widetilde{P} g(x) \right\|^2_{\widetilde{H}} \Big] dx \\
&\overset{3.1.6\ 2.}{\leqslant} C \int_0^s \Big[\left\| \widetilde{S}_1(x) \left(\mathbb{1} - Q_{-\frac{1}{2}} \right) \right\|^2_{\mathcal{L}(\widetilde{H})} x \\
&\qquad + \left\| \widetilde{S}_1(x) Q_{-\frac{1}{2}} \right\|^2_{\mathcal{L}(\widetilde{H})} x|\log x| \Big] dx \, \|g\|^2_{L^2(I,H)} \\
&=: K_2(s) \, \|g\|^2_{L^2(I,H)} ,
\end{aligned}$$

where $K_2(s) \overset{s\to 0}{\to} 0$ by Definition 2.1.1, 7. Thus, for s sufficiently small we have

$$\left\| (B + S_1) \,\widehat{\mathrm{P}} \right\|_{\mathcal{L}(L^2((0,s],H_{\mathrm{tot}}))} < 1. \tag{3.16}$$

Let us fix such an s. Then we obtain for $\varphi \in C_0^\infty([0,s))$

$$\varphi u = \widehat{\mathrm{P}} \left(\sum_{j=0}^\infty (-1)^j \left([B + S_1] \,\widehat{\mathrm{P}} \right)^j \right) D_{\max} \varphi u =: \widehat{\mathrm{P}}\, V D_{\max} \varphi u$$

with V bounded. $\square$

Lemma 3.2.4. [9] *Let D be an operator with horn singularity. Then $\mathcal{D}(D_{\min}) \subset \mathcal{D}_1$ and $\mathcal{D}((D_c)_{\min}) \subset \mathcal{D}_1^c$. If $-\frac{1}{2} \notin \operatorname{spec} \widetilde{S}$, then all these sets are equal.*

Proof. We prove the assertion for $D_{\min}$. The proof for $(D_c)_{\min}$ is almost identical (actually, it is more simple).

First, we show that $\mathcal{D}_1$ is closed:
Let $\widehat{u}_n \in \mathcal{D}_1$ with $\widehat{u}_n \overset{n\to\infty}{\longrightarrow} \widehat{u}$. Since $D_{\max}$ is closed, this implies $\widehat{u} \in \mathcal{D}(D_{\max})$ and $\lim_{n\to\infty} D_{\max}\widehat{u}_n = D_{\max}\widehat{u}$. Since $\widehat{P}$ and V are bounded operator, we find for $\varphi \in C_0^\infty([0,s))$

$$\varphi\widehat{u} = \lim_{n\to\infty} \varphi\widehat{u}_n \overset{(3.14)}{=} \lim_{n\to\infty} \widehat{\mathrm{P}}\, V D_{\max}\varphi\widehat{u}_n = \widehat{\mathrm{P}}\, V D_{\max}\varphi\widehat{u}.$$

This implies for $\widehat{u}|_U = u + \widetilde{u}$

$$\begin{aligned}
\|\varphi(x)u(x)\|_H &= \|U(x)P_> \oplus P_< \left[U^* V D_{\max}\varphi u\right](x)\|_H \\
&\overset{3.1.6,\ 2.}{\leqslant} Cx^{\frac{\beta}{2}} \|V D_{\max}\varphi u\|_{L^2(I,H)} \\
\left\|\left(\mathbb{1} - Q_{-\frac{1}{2}}\right)\varphi(x)\widetilde{u}(x)\right\|_{\widetilde{H}} &= \left\|\left(\mathbb{1} - Q_{-\frac{1}{2}}\right)\widetilde{P}\left[U^* V D_{\max}\varphi\widetilde{u}\right](x)\right\|_{\widetilde{H}}, \\
&\overset{3.1.6,\ 2.}{\leqslant} Cx^{\frac{1}{2}} \|V D_{\max}\varphi\widetilde{u}\|_{L^2(I,\widetilde{H})} \\
\left\|Q_{-\frac{1}{2}}\varphi(x)\widetilde{u}(x)\right\|_{\widetilde{H}} &= \left\|Q_{-\frac{1}{2}}\widetilde{P}\left[U^* V D_{\max}\varphi\widetilde{u}\right](x)\right\|_{\widetilde{H}}, \qquad \text{and} \\
&\overset{3.1.6,\ 2.}{\leqslant} Cx^{\frac{1}{2}}|\log x|^{\frac{1}{2}} \|V D_{\max}\varphi\widetilde{u}\|_{L^2(I,\widetilde{H})}.
\end{aligned}$$

Thus, $\widehat{u} \in \mathcal{D}_1$ and $\mathcal{D}_1$ is closed. Since $D_{\min}$ is the closure of D on $C_0^\infty(M,E)$, this implies $\mathcal{D}(D_{\min}) \subset \mathcal{D}_1$.

We assume that $-\frac{1}{2} \notin \operatorname{spec} \widetilde{S}$ and prove $\mathcal{D}_1 \subset \mathcal{D}(D_{\min})$. Let $\widehat{u} \in \mathcal{D}_1$, i.e. $\widehat{u} = u + \widetilde{u} \in \mathcal{D}(D_{\max})$ with $\|u(x)\|_H = O(x^{\frac{\beta}{2}})$ and $\|\widetilde{u}(x)\|_{\widetilde{H}} = O(x^{\frac{1}{2}})$ as $x \to 0$. Let $\varphi, \chi \in C^\infty((0,s_0),[0,1])$ with

$$\varphi(x) = \begin{cases} 0 & x \leqslant \frac{1}{2} \\ 1 & x \geqslant 1 \end{cases} \text{ and } \varphi'(x) \leqslant 3,\ \chi(x) := \begin{cases} x & 0 \leqslant x \leqslant \frac{1}{2} \\ 1 & x \geqslant 1 \end{cases} \text{ and } \chi'(x) \leqslant 2$$

[9] This lemma is similar to Lemma 3.4 in [Brü92] that is partially a restatement of Theorem 6.1 in [BS87].

and extend them to the whole manifold M by setting their value equal to one on M_1. Set $\varphi_n(x) := \varphi(nx)$ and $\psi_n(x) := \chi(x)^{\alpha_n}\varphi_n(x)$ where $\alpha_n := [\log n]^{-\frac{1}{2}}$ for $n \geqslant 2$.

$$\begin{aligned}
\int_0^{s_0} \left[\varphi_n'(x)\right]^2 \|u(x)\|_H^2\, dx &\leqslant \int_0^{s_0} n^2(\varphi'(nx))^2 C^2 x^\beta dx \leqslant 9n^2C^2 \int_{\frac{1}{2n}}^{\frac{1}{n}} x^\beta dx \\
&\leqslant \frac{9C^2}{\beta+1} n^{1-\beta} \overset{n\to\infty}{\longrightarrow} 0 \quad \text{ since } \beta > 1 \\
\int_0^{s_0} \left[\psi_n'(x)\right]^2 \|\widetilde{u}(x)\|_{\widetilde{H}}^2\, dx &\leqslant \int_0^{s_0} \left[\alpha_n \chi(x)^{\alpha_n - 1}\chi'(x)\varphi(nx) + \chi(x)^{\alpha_n} n\varphi'(nx)\right]^2 C^2 x dx \\
&\leqslant 8C^2\alpha_n^2 \int_0^1 x^{2\alpha_n - 1} dx + 18C^2n^2 \int_{\frac{1}{2n}}^{\frac{1}{n}} x^{2\alpha_n+1} dx \\
&\leqslant 4C^2\alpha_n + \frac{18C^2}{2\alpha_n+2} n^{-2\alpha_n} \leqslant 4C^2\alpha_n + 9C^2 e^{-2[\log n]^{\frac{1}{2}}} \overset{n\to\infty}{\longrightarrow} 0 \quad \text{ since } \alpha_n \to 0
\end{aligned}$$

Let $\widehat{v}_n$ be defined by $\widehat{v}_n|_U := \varphi_n u + \psi_n \widetilde{u}$ and $\widehat{v}_n|_{M_1} := \widehat{u}$. Since φ_n and ψ_n have compact support, it is clear that $\widehat{v}_n \in \mathcal{D}(D_{\min})$ for all $n \in \mathbb{N}$ and

$$\|\widehat{u} - \widehat{v}_n\|_{L^2(M,E)}^2 \leqslant 2\int_0^{\frac{1}{n}} \|u(x)\|_H^2 + \|\widetilde{u}(x)\|_{\widetilde{H}}^2\, dx \overset{n\to\infty}{\longrightarrow} 0.$$

Let wlog $n \leqslant m$:

$$\begin{aligned}
&\|D_{\min}(\widehat{v}_n - \widehat{v}_m)\|_{L^2(M,F)}^2 \\
&\quad\leqslant 4\int_0^{s_0} \left\|(\varphi_n' - \varphi_m')(x)u(x)\right\|_H^2 dx + 4\left\|(\varphi_n - \varphi_m)D_{\max}u\right\|_{L^2(I,H)}^2 \\
&\qquad + 4\int_0^{s_0} \left\|(\psi_n' - \psi_m')(x)\widetilde{u}(x)\right\|_{\widetilde{H}}^2 dx + 4\left\|(\psi_n - \psi_m)D_{\max}\widetilde{u}\right\|_{L^2(I,\widetilde{H})}^2 \\
&\quad\leqslant 8\int_0^{s_0} \left(\left[\varphi_n'(x)\right]^2 + \left[\varphi_m'(x)\right]^2\right) \|u(x)\|_H^2\, dx + 4\int_{\frac{1}{2m}}^{\frac{1}{n}} \|D_{\max}u(x)\|_H^2\, dx \\
&\qquad + 8\int_0^{s_0} \left(\left[\psi_n'(x)\right]^2 + \left[\psi_m'(x)\right]^2\right) \|\widetilde{u}(x)\|_{\widetilde{H}}^2\, dx + 4\int_{\frac{1}{2m}}^{\frac{1}{n}} \|D_{\max}\widetilde{u}(x)\|_{\widetilde{H}}^2\, dx \\
&\quad\overset{n,m\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Thus, we have shown that $D_{\min}\widehat{v}_n$ is a Cauchy sequence in $L^2(M,F)$. Therefore, there is a $\widehat{v} \in L^2(M,F)$, such that $D_{\min}\widehat{v}_n \to \widehat{v}$. Since we have already proved that $\widehat{v}_n \in \mathcal{D}(D_{\min})$, $\widehat{v}_n \to \widehat{u}$ and since $D_{\min}$ is closed, we find that $\widehat{u} \in \mathcal{D}(D_{\min})$.

In summary,

$$\mathcal{D}(D_{\min}) = \mathcal{D}_1 \overset{3.2.3}{\cong} \mathcal{D}_1^c = \mathcal{D}((D_c)_{\min})$$

□

Remark: Arbitrarily close to the singularity, the manifold M can be perturbed to a product and can be doubled. Then D extends to an elliptic differential operator of first order over this compact manifold $\widetilde{M}$. As such it has an interior parametrix $P_i \in \mathcal{L}(L^2(\widetilde{M},\widetilde{F}), H^1(\widetilde{M},\widetilde{E}))$, such that

$$DP_i = \mathbb{1} - R \qquad \text{and} \qquad P_iD = \mathbb{1} - L,$$

where $R \in \mathcal{L}(L^2(\widetilde{M},\widetilde{F}))$ and $L \in \mathcal{L}(H^1(\widetilde{M},\widetilde{E}))$ are compact operators (Theorem 4.6 on page 191 in [LM89]). For all $\varphi, \psi \in C^\infty(M,[0,\infty))$ with $\varphi \equiv 1$ and $\psi \equiv 0$ on the perturbed neighbourhood of the singularity and $\psi(1-\varphi) = (1-\varphi)$ follows

$$\begin{aligned} DP_\varphi &:= D\psi P_i(1-\varphi) = \psi' P_i(1-\varphi) + (1-\varphi) - \psi R(1-\varphi) =: 1 - \varphi + R_\varphi \\ P_\varphi D &= \psi P_i(1-\varphi)D = \psi P_i D(1-\varphi) + \psi P_i \varphi' = (1-\varphi) - \psi L(1-\varphi) + \psi P_i \varphi' \\ &=: 1 - \varphi + L_\varphi, \end{aligned}$$

where $L_\varphi \in \mathcal{L}(L^2(M,E))$, $R_\varphi \in \mathcal{L}(H^1(M,F))$ are compact and $P_\varphi \in \mathcal{L}(L^2(M,F), H^1(M,E))$. The formal adjoint D^t has an equivalent property.

Theorem 3.2.5. [10] *Let D be an operator with horn singularity. $D_{\min}$ and $D_{\max}$ are Fredholm operators. $\mathcal{D}(D_{\max})/\mathcal{D}(D_{\min})$ is finite-dimensional. The closed extensions of $D_{\min}$ are defined by*

$$W \subset \mathcal{D}(D_{\max})/\mathcal{D}(D_{\min}), \quad \mathcal{D}(D_W) := \mathcal{D}(D_{\min}) \oplus W, \quad D_W := D_{\max}|_{\mathcal{D}(D_W)}.$$

They are all Fredholm, and their indices are given by

$$\operatorname{ind} D_W = \operatorname{ind} D_{\min} + \dim W.$$

Furthermore,

$$\operatorname{ind} D_{\min} = \operatorname{ind}(D_c)_{\min}.$$

Proof. We define

$$P_{\min} := \psi\, \widehat{\mathrm{P}}\, \varphi + P_\varphi,$$

where $\varphi \in C_0^\infty([0,s),[0,1])$ with s as in Formula (3.14) and $\psi \in C_0^\infty([0,s_0),[0,1])$ with $\varphi = 1$ near 0 and $\psi = 1$ near $\operatorname{supp}\varphi$. We apply the decomposition (3.15)

$$D = UD_cU^* + B + S_1$$

and compute

$$\begin{aligned} D_{\min}P_{\min} &= (U(D_c)_{\min}U^* + B + S_1)\,(\psi UPU^*\varphi) + D_{\min}P_\varphi \\ &= \psi'\,\widehat{\mathrm{P}}\,\varphi + \underbrace{\psi U(D_c)_{\min}PU^*\varphi}_{=\psi\varphi=\varphi} + \psi(B+S_1)\,\widehat{\mathrm{P}}\,\varphi + (1-\varphi)\mathbb{1} + R_\varphi \\ &= \mathbb{1} + \left(\psi'\,\widehat{\mathrm{P}}\,\varphi + R_\varphi\right) + \psi(B+S_1)\,\widehat{\mathrm{P}}\,\varphi =: \mathbb{1} + \widetilde{K} + R \end{aligned}$$

[10] This theorem is similar to Theorem 3.2 in [Brü92] which is stated for the cone case.

where $\widetilde{K}$ is compact (3.1.6, 4.) and $\|R\| < 1$ by Formula (3.16). This implies we have found a parametrix:

$$D_{\min} \underbrace{P_{\min}(\mathbb{1} - R)^{-1}}_{=:P_1} = \mathbb{1} + \widetilde{K}(\mathbb{1} - R)^{-1} =: \mathbb{1} + K.$$

and

$$P_1^*(D_{\min})^* = \mathbb{1} + K^* \quad \Rightarrow \quad \dim\operatorname{coker} D_{\min} = \dim\ker(D_{\min})^* < \infty.$$

Since $D_{\min} \subset D_{\max}$, this implies also

$$\dim\operatorname{coker} D_{\max} < \infty.$$

If D is an operator with horn singularity, then the formal adjoint D^t with

$$D^t|_U \cong \left(-\frac{\partial}{\partial x} + x^{-\beta}S(x) + S_1(x)\right) \oplus \left(-\frac{\partial}{\partial x} + x^{-1}\widetilde{S} + \widetilde{S}_1(x)\right)$$

is also an operator with horn singularity (Note that for D^t the eigenvalue $\frac{1}{2}$ plays the special role that $-\frac{1}{2}$ played for D). Therefore, the argumentation above shows that there are a parametrix P_2 and a compact operator K', such that

$$D^t_{\min}P_2 = \mathbb{1} + K'$$

and

$$\dim\ker D_{\max} = \dim\ker(D^t_{\min})^* < \infty \quad \text{and} \quad \dim\ker D_{\min} < \infty.$$

Due to Lemma 19.1.1 in [Hö85] (page 181), this proves that $D_{\min}$, $D_{\max}$ and all closed extensions in between are Fredholm operators.

$$\begin{aligned}
\dim\left(\mathcal{D}(D_{\max})\big/\mathcal{D}(D_{\min})\right) &= \dim\left(\ker D_{\max}\big/\ker D_{\min}\right) + \dim\left(\operatorname{im} D_{\max}\big/\operatorname{im} D_{\min}\right)\\
&= \dim\left(\ker D_{\max}\big/\ker D_{\min}\right) + \dim\left((\operatorname{im} D_{\min})^{\perp}\big/(\operatorname{im} D_{\max})^{\perp}\right)\\
&= \dim\ker D_{\max} - \dim\ker D_{\min} + \dim\operatorname{coker} D_{\min} - \dim\operatorname{coker} D_{\max}\\
&= \operatorname{ind} D_{\max} - \operatorname{ind} D_{\min}
\end{aligned}$$

Now, for $W \subset \mathcal{D}(D_{\max})\big/\mathcal{D}(D_{\min})$ we define the closed operator

$$\mathcal{D}(D_W) := \mathcal{D}(D_{\min}) \oplus W, \quad D_W := D_{\max}|_{\mathcal{D}(D_W)}.$$

To compute the index, we decompose

$$W = (W \cap \ker D_W) \oplus \underbrace{(W \cap (\ker D_W)^{\perp})}_{\cong D_W(W)}$$

and calculate the index

$$\begin{aligned}
\operatorname{ind} D_W &= \dim\ker D_{\min} + \dim(W \cap \ker D_W) - (\dim\operatorname{coker} D_{\min} - \dim D_W(W))\\
&= \operatorname{ind} D_{\min} + \dim W.
\end{aligned}$$

Choose $s \in (0, s_0]$ as in Formula (3.14), $s_1 \in (0, s)$ and $\varphi \in C_0^\infty([0,s),[0,1])$ with $\varphi|_{[0,s_1]} = 1$. Let $u \in \mathcal{D}(D_{\min})$.

$$\begin{aligned}
&\|[B+S_1]u\|^2_{L^2(I,H_{\text{tot}})} \leqslant 2\left[\|[B+S_1]\varphi u\|^2_{L^2((0,s],H_{\text{tot}})} + \|[B+S_1](1-\varphi)u\|^2_{L^2((s_1,s_0],H_{\text{tot}})}\right] \\
&\overset{(3.14)}{=} 2\left\|[B+S_1]\,\widehat{\mathrm{P}}\,VD\varphi u\right\|^2_{L^2((0,s],H_{\text{tot}})} + 2\int_{s_1}^{s_0} \|[B+S_1](x)\underbrace{(1-\varphi(x))}_{\leqslant 1} u(x)\|^2 dx \\
&\leqslant 2\underbrace{\left\|[B+S_1]\,\widehat{\mathrm{P}}\right\|^2_{\mathcal{L}(L^2((0,s],H_{\text{tot}}))}}_{\overset{(3.16)}{<}1} \|V\|^2_{\mathcal{L}(L^2((0,s],H_{\text{tot}}))} \\
&\qquad\qquad \cdot 2\left(\|\varphi'\|^2_\infty \|u\|^2_{L^2((0,s],H_{\text{tot}})} + \underbrace{\|\varphi\|^2_\infty}_{\leqslant 1} \|Du\|^2_{L^2((0,s],H_{\text{tot}})}\right) \\
&\quad + 2\sup_{x\in[s_1,s_0]} \|[B+S_1](x)\|\,\|u\|^2_{L^2((s_1,s_0],H_{\text{tot}})} \leqslant C\,\|u\|_{\mathcal{D}(D)}
\end{aligned}$$

Therefore, $B + S_1 : \mathcal{D}(D_{\min}) \to L^2(M,E)$ is a bounded operator and

$$[0,1] \ni a \longmapsto (UD_cU^* + a\,[B+S_1])$$

is a continuous path in $\mathcal{L}(\mathcal{D}(D_{\min}), L^2(M,H_{\text{tot}}))$. Due to Theorem 19.1.5 on page 182 in [Hö85], $(D_c)_{\min}$ is also Fredholm and $\operatorname{ind} D_{\min} = \operatorname{ind}(D_c)_{\min}$. $\square$

3.3 An index preserving homotopy

We define the homotopy

$$[0,1] \ni r \mapsto T_r := \frac{\partial}{\partial x} + x^{-\beta} S_c\Big(\underbrace{\left((1-r)x^{1-\beta} + rs_0^{1-\beta}\right)^{\frac{1}{1-\beta}}}_{=:\xi_r(x)}\Big) =: \frac{\partial}{\partial x} + x^{-\beta} S_r(x)$$

that connects

$$T_0 = \frac{\partial}{\partial x} + x^{-\beta} S_c(x) \quad \text{and} \quad T_1 := \frac{\partial}{\partial x} + x^{-\beta} S_c(s_0).$$

For a given operator with horn singularity D, we define the homotopy $[0,1] \ni r \mapsto D_r$ by $D_r|_{M_1} := D|_{M_1}$ and $D_r|_U \cong T_r \oplus \widetilde{T}$. In this section we will prove that if D is an operator with C^1-horn singularity, then each D_r is an operator with C^1-horn singularity with index

$$\operatorname{ind}(D_r)_{\min} = \operatorname{ind}(D_0)_{\min} \overset{\text{Theorem 3.2.5}}{=} \operatorname{ind} D_{\min}.$$

To ease notation, we omit the subscript c.

Let $r \in [0,1]$ and $x \in (0, s_0]$

$$\xi_0(x) = x, \quad \xi_1(x) \equiv s_0$$

$$\xi_r(s_0) = \left((1-r){s_0}^{1-\beta} + r s_0^{1-\beta}\right)^{\frac{1}{1-\beta}} \equiv s_0 \tag{3.17}$$

$$\frac{\partial \xi_r}{\partial x}(x) = \frac{1}{1-\beta}\left((1-r)x^{1-\beta} + r s_0^{1-\beta}\right)^{\frac{1}{1-\beta}-1}(1-r)\,(1-\beta)x^{-\beta} = (\xi_r(x))^{\beta}\,(1-r)\,x^{-\beta} \tag{3.18}$$

$$\frac{\partial \xi_r}{\partial r}(x) = \frac{1}{1-\beta}\left((1-r)x^{1-\beta} + r s_0^{1-\beta}\right)^{\frac{1}{1-\beta}-1}\left(-x^{1-\beta} + s_0^{1-\beta}\right) = (\xi_r(x))^{\beta}\,\frac{x^{1-\beta} - s_0^{1-\beta}}{\beta - 1}$$

$$\begin{aligned}\frac{\partial^2 \xi_r}{\partial r \partial x}(x) &= \frac{\partial}{\partial r}\left((\xi_r(x))^{\beta}(1-r)x^{-\beta}\right) = \beta(\xi_r(x))^{\beta-1}\frac{\partial \xi_r}{\partial r}(x)(1-r)x^{-\beta} - \xi_r(x)^{\beta}x^{-\beta} \\ &= \frac{\beta}{\beta-1}(\xi_r(x))^{2\beta-1}(1-r)(x^{1-2\beta} - x^{-\beta}s_0^{1-\beta}) - \xi_r(x)^{\beta}x^{-\beta}\end{aligned}$$

Let $r \in [0,1]$ and $x \in [\delta, s_0]$ for a $\delta > 0$:

$$\left|\frac{\partial \xi_r}{\partial x}(x)\right| \leqslant s_0^{\beta}\delta^{-\beta}, \quad \left|\frac{\partial \xi_r}{\partial r}(x)\right| \leqslant \frac{s_0^{\beta}}{\beta-1}\delta^{1-\beta}, \quad \left|\frac{\partial^2 \xi_r}{\partial r \partial x}(x)\right| \leqslant \frac{\beta}{\beta-1}s_0^{2\beta-1}\delta^{1-2\beta} \tag{3.19}$$

Let $r \in [0,1)$ and $x \in (0, s_0]$:

$$\lim_{x\to 0}\xi_r(x) = \lim_{x\to 0} x\left((1-r) + r\left(\frac{x}{s_0}\right)^{\beta-1}\right)^{\frac{1}{1-\beta}} \overset{\beta \geq 1}{=} 0 \tag{3.20}$$

Let $r \in [0,1)$ and $y \in (0, s_0]$:

$$\xi_r^{-1}(y) = \left(\frac{y^{1-\beta} - r s_0^{1-\beta}}{1-r}\right)^{\frac{1}{1-\beta}}$$

$$\frac{\partial \xi_r^{-1}}{\partial y}(y) = \frac{1}{1-\beta}\left(\frac{y^{1-\beta} - r s_0^{1-\beta}}{1-r}\right)^{\frac{\beta}{1-\beta}}\frac{(1-\beta)y^{-\beta}}{1-r} = \left(\xi_r^{-1}(y)\right)^{\beta}\frac{y^{-\beta}}{1-r}$$

Lemma 3.3.1. [11] *If D is an operator with (C^1-)horn singularity, then D_r is also an operator with (C^1-)horn singularity for all $r \in [0,1]$.*

Proof. By construction the operators D_r differ from $D_c = D_0$ only in the T_r part. Thus, the assumptions referring to M_1 and $\widetilde{T}$ are automatically fulfilled. Furthermore, we find by setting $Q_{r>}(x) := Q_>(\xi_r(x))$ and $Q_{r<}(x) := Q_<(\xi_r(x))$:

$$\begin{aligned} S_{r>}(x) &:= S_>(\xi_r(x)) = S_r(x)Q_{r>}(x) \geqslant C_2 Q_{r>}(x) \qquad \text{and} \\ S_{r<}(x) &:= S_<(\xi_r(x)) = S_r(x)Q_{r<}(x) \leqslant -C_2 Q_{r<}(x)\end{aligned}$$

[11] This lemma is similar to the first part of Theorem 4.1 in [Brü92] which is stated for the cone case.

for all $x \in (0, s_0]$, since $\xi_r(x) \in (0, s_0]$ for $x \in (0, s_0]$.

$$\alpha_r(x) := \left\| S_r'(x) S_r(x)^{-1} \right\|_{\mathcal{L}(H)} = \frac{\partial \xi_r}{\partial x}(x) \alpha\left(\xi_r(x)\right)$$

In particular, since $\xi_1 \equiv s_0$, it follows that $\alpha_1 \equiv 0$. Let $r \in [0, 1)$. In the HO^2 case we find

$$\begin{aligned}
\int_0^{s_0} \alpha_r^2(x) x^\beta dx &= \int_0^{s_0} \alpha^2(\xi_r(x)) \left(\frac{\partial \xi_r}{\partial x}(x)\right)^2 x^\beta dx \\
&\overset{(3.18)}{=} \int_0^{s_0} \alpha^2(\xi_r(x))(\xi_r(x))^\beta (1-r) \frac{\partial \xi_r}{\partial x}(x) dx \\
&= \int_{\xi_r(0)}^{\xi_r(s_0)} \alpha^2(y) y^\beta (1-r) dy \overset{(3.17),\,(3.20)}{=} (1-r) \int_0^{s_0} \alpha^2(y) y^\beta dy \\
&\leqslant \int_0^{s_0} \alpha^2(y) y^\beta dy = C_4^2(s_0).
\end{aligned}$$

In the HO$^\infty$ case we find

$$\begin{aligned}
\sup_{x \in (0,s]} x^\beta \alpha_r(x) &= \sup_{x \in (0,s]} x^\beta \frac{\partial \xi_r}{\partial x}(x) \alpha(\xi_r(x)) \overset{(3.18)}{=} \sup_{x \in (0,s]} x^\beta (\xi_r(x))^\beta (1-r) x^{-\beta} \alpha(\xi_r(x)) \\
&= (1-r) \sup_{x \in (0, \xi_r(s)]} x^\beta \alpha(x) = (1-r) C_4(\xi_r(s)) \xrightarrow{s \to 0} 0
\end{aligned}$$

since by Formula (3.20), $\xi_r(s) \xrightarrow{s \to 0} 0$.

Thus, D_r is an operator with horn singularity for all $r \in [0, 1]$. The regularity assertion follows from the regularity of ξ_r. □

We want to apply the following lemma proved as Lemma 4.1.1 on page 55.

Lemma 3.3.2. *Let $D_r : \mathcal{H}_1 \supset \mathcal{D}(D_r) \to \mathcal{H}_2$ be a one-parameter family of densely defined closed Fredholm operators of Hilbert spaces. We define the closed operator*

$$E_r := \begin{pmatrix} \mathbb{1} & -D_r^* \\ D_r & \mathbb{1} \end{pmatrix} : \mathcal{D}(D_r) \oplus \mathcal{D}(D_r^*) \to \mathcal{H}_1 \oplus \mathcal{H}_2.$$

Then E_r^{-1} is a bounded operator and if the map

$$r \mapsto E_r^{-1} \in \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2)$$

is continuous, $\operatorname{ind} D_r$ *is independent of r.*

If we take $\mathcal{H}_1 = L^2(M, E)$, $\mathcal{H}_2 = L^2(M, F)$ and $D_r = (D_r)_{\min}$, then the lemma reduces our problem to showing that $r \mapsto E_r^{-1}$ is continuous. Let φ, $\psi \in C_0^\infty([0, s_0))$ with $\varphi|_{[0,\delta]} = 1$, $\psi|_{\operatorname{supp} \varphi} = 1$ and $\psi R_\varphi = 0 = \psi L_\varphi^*$ and define

$$F_r := \begin{pmatrix} 0 & \psi P_r \varphi + P_\varphi \\ -\varphi P_r^* \psi - P_\varphi^* & 0 \end{pmatrix},$$

where P_r is a parametrix to $(D_r)_{\min}$. The existence of these parametrices follows from Lemma 3.2.3 since all D_r are operators with horn singularities. F_r is a parametrix for E_r:

$$\begin{aligned}
E_rF_r &= \begin{pmatrix} \mathbb{1} & -(D_r)^*_{\min} \\ (D_r)_{\min} & \mathbb{1} \end{pmatrix} \begin{pmatrix} 0 & \psi P_r\varphi + P_\varphi \\ -\varphi P_r^*\psi - P_\varphi^* & 0 \end{pmatrix} \\
&= \begin{pmatrix} (D_r)^*_{\min}(\varphi P_r^*\psi + P_\varphi^*) & \psi P_r\varphi + P_\varphi \\ -\varphi P_r^*\psi - P_\varphi^* & (D_r)_{\min}(\psi P_r\varphi + P_\varphi) \end{pmatrix} \\
&= \begin{pmatrix} \mathbb{1} + \varphi' P_r^*\psi + L_\varphi^* & \psi P_r\varphi + P_\varphi \\ -\varphi P_r^*\psi - P_\varphi^* & \mathbb{1} + \psi' P_r\varphi + R_\varphi \end{pmatrix} \\
&= \mathbb{1} + \begin{pmatrix} 0 & \psi P_r\varphi + P_\varphi \\ -\varphi P_r^*\psi - P_\varphi^* & 0 \end{pmatrix} + \begin{pmatrix} \varphi' P_r^*\psi + L_\varphi^* & 0 \\ 0 & \psi' P_r\varphi + R_\varphi \end{pmatrix} =: \mathbb{1} + F_r + G_r.
\end{aligned}$$

We gather some properties of the operators defined above in the following lemma.

Lemma 3.3.3. [12] *Let Z_r denote the orthogonal projection onto $\ker F_r$.*

1. *E_rF_r is Fredholm with index 0.*
2. *Z_r is of finite rank and hence compact.*
3. *$G_r : L^2(M,E) \oplus L^2(M,F) \to \mathcal{D}(D_\gamma) \oplus \mathcal{D}(D_\gamma^*) = \mathcal{D}(E_\gamma)$ for all $\gamma \in [0,1]$.*
4. *$F_r + Z_r$ is bijective.*
5. *$Z_r = -G_rZ_r$ and $\operatorname{im} Z_r \subset \mathcal{D}(E_\gamma)$ for all r and $\gamma \in [0,1]$.*

Proof.

1. By definition and Theorem 3.1.6, 4. the operator $F_r + G_r$ is compact and thus, $\mathbb{1} + F_r + G_r = E_rF_r$ is Fredholm with index $\operatorname{ind}(E_rF_r) = \operatorname{ind} \mathbb{1} = 0$.
2. From 1. we know that E_rF_r has a finite-dimensional kernel. Thus, F_r has a finite-dimensional kernel.
3. $(D_r)_{\min}$ and $(D_r)^*_{\min}$ are closed extensions of first order elliptic differential operators for all $r \in [0,1]$. Whether a function belongs to $\mathcal{D}((D_r)_{\min})$ or $\mathcal{D}((D_r)^*_{\min})$, depends on its asymptotic behaviour at the singularity. From the definition, we see that there is a neighbourhood of the singularity V, such that $G_rf|_V = 0$ for all $f \in L^2(M, E \oplus F)$.
4. The operator F_r has the following property

$$F_r^* = \begin{pmatrix} 0 & \psi P_r\varphi + P_\varphi \\ -\varphi P_r^*\psi - P_\varphi^* & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & -\psi P_r\varphi - P_\varphi \\ \varphi P_r^*\psi + P_\varphi^* & 0 \end{pmatrix} = -F_r.$$

$$\Rightarrow \operatorname{im} F_r = (\ker F_r^*)^\perp = (\ker F_r)^\perp$$

$$\Rightarrow \begin{cases} F_r|_{(\ker F_r)^\perp} : (\ker F_r)^\perp \to (\ker F_r)^\perp & \text{is bijective.} \\ Z_r|_{\ker F_r} : \ker F_r \to \ker F_r & \text{is the identical map and thus bijective.} \end{cases}$$

$\Rightarrow F_r + Z_r$ is bijective.

[12] This lemma is a restatement of Lemma 4.7 in [LP98] which refers to the proof of Theorem 4.1 in [BS88]. We added more details to the statement and the proof.

5. We compute

$$\begin{aligned} && E_r F_r &= \mathbb{1} + F_r + G_r && | \circ Z_r \\ \Rightarrow && 0 &= Z_r + 0 + G_r Z_r && \\ \Rightarrow && Z_r &= -G_r Z_r. && \end{aligned}$$

From 3. follows $\operatorname{im} Z_r \subset \operatorname{im} G_r \subset D(E_\gamma)$ for all r and $\gamma \in [0,1]$.

□

If the map $r \mapsto P_r$ is continuous, then

$$r \mapsto H_r := E_r(F_r + Z_{r_0}) = \mathbb{1} + F_r + G_r + E_r Z_{r_0}$$

is also continuous in r for every $r_0 \in [0,1]$ since S_r is strongly continuously differentiable (which implies by Lemma 3.5 in [Kre71] page 11 that $E_r Z_{r_0}$ is continuous in norm). Furthermore, since the right side is the sum of the identity and a compact operator, H_r is a family of Fredholm operators with $\operatorname{ind} H_r = \operatorname{ind} \mathbb{1} = 0$. From Lemma 3.3.3, 4. follows that $H_{r_0} = E_{r_0}(F_{r_0} + Z_{r_0})$ is invertible. Due to the continuity of the family, H_r is invertible near r_0. Hence

$$r \mapsto E_r^{-1} = (F_r + Z_{r_0}) \left(E_r(F_r + Z_{r_0})\right)^{-1} = (F_r + Z_{r_0}) H_r^{-1}$$

is continuous and thus, Lemma 3.3.2 can be applied, i.e. we have proved

Theorem 3.3.4. *Let D be an operator with C^1-horn singularity. Then*

$$\operatorname{ind}(D_r)_{\min}$$

is independent of r.

It remains to prove that $r \mapsto P_r$ is continuous.

Lemma 3.3.5. [13] *Let S be strongly differentiable and let $x \mapsto S(x)S(s_0)^{-1}$ be norm continuous. Denote by $W_{r>}(x,y)$ and $W_{r<}(x,y)$ the solution operators of $S_{r>}(x) = S_>(\xi_r(x))$ and $S_{r<}(x) = S_<(\xi_r(x))$, respectively. Let $a \in (0,1)$ and $s_1 \in (0, s_0)$. For every $\varepsilon > 0$ exists a constant $C_\varepsilon > 0$ and a $\delta > 0$, such that*

$$\begin{aligned} &\|W_{r>}(x,y) - W_{t>}(x,y)\|_{\mathcal{L}(Q_> H)} + \|W_{r<}(y,x) - W_{t<}(y,x)\|_{\mathcal{L}(Q_< H)} \\ &\qquad \leqslant C_\varepsilon e^{-dF(x,y)} \left(F(x,y)^{-a} + 1\right) \varepsilon \end{aligned} \tag{3.21}$$

for all $s_1 \leqslant y < x \leqslant s_0$ and $|r - t| \leqslant \delta$.

Proof. The assumptions on S imply by Lemma 2.2.2 that $Q_>$ and $Q_<$ are continuously differentiable in norm. That implies $x \mapsto S'_>(x)S_>(s_0)^{-1}$ and $x \mapsto S'_<(x)S_<(s_0)^{-1}$ are norm

[13] In the cone case this lemma was a remark in the proof of Lemma 3.2 on page 286 in [Brü92]. The full proof given here shows that it suffices to assume that S is strongly differentiable and $x \mapsto S'(x)S(s_0)^{-1}$ is continuous in norm. This is slightly more than what had to be assumed to prove the Fredholm property.

continuous. Let $a \in (0,1)$, $s_1 \in (0, s_0)$ and $\varepsilon > 0$ be fixed. Let $x \in [s_1, s_0]$ and $r, t \in [0,1]$. We estimate

$$\begin{aligned}
&\left\|(S_{r>}(x)-\zeta)^{-1}-(S_{t>}(x)-\zeta)^{-1}\right\| = \left\|\int_t^r \frac{\partial}{\partial u}(S_>(\xi_u(x))-\zeta)^{-1}du\right\| \\
&\quad = \left\|\int_t^r (-1)(S_>(\xi_u(x))-\zeta)^{-1}S'_>(\xi_u(x))\frac{\partial \xi_u(x)}{\partial u}(S_>(\xi_u(x))-\zeta)^{-1}du\right\| \\
&\quad \leqslant C_{s_1}|\zeta|^{-1}\sup_{x\in[s_1,s_0],\ u\in[0,1]}\left|\frac{\partial \xi_u(x)}{\partial u}\right|\sup_{x\in[s_1,s_0]}\left\|S'_>(x)S_>(x)^{-1}\right\|\left|\int_t^r du\right| \\
&\quad \overset{(3.19)}{\leqslant} C_{s_1}|\zeta|^{-1}|r-t|.
\end{aligned} \tag{3.22}$$

As in the proof of Lemma 3.1.2 we approximate $W_{r>}(x,y)$ by

$$\widetilde{W}_{r>}(x,y) := \frac{1}{2\pi i}\int_c e^{-\zeta F(x,y)}(S_{r>}(x)-\zeta)^{-1}d\zeta$$

and calculate

$$\begin{aligned}
\left\|\widetilde{W}_{r>}(x,y)-\widetilde{W}_{t>}(x,y)\right\| &\leqslant \left\|\frac{1}{2\pi}\int_c e^{-\zeta F(x,y)}\left[(S_{r>}(x)-\zeta)^{-1}-(S_{t>}(x)-\zeta)^{-1}\right]d\zeta\right\| \\
&\leqslant Ce^{-dF(x,y)}\int_c e^{-|\zeta-d|F(x,y)}\left\|(S_{r>}(x)-\zeta)^{-1}-(S_{t>}(x)-\zeta)^{-1}\right\|d\zeta \\
&\overset{(3.22)}{\leqslant} C_{s_1}e^{-dF(x,y)}F(x,y)^{-a}|r-t|\int_c|\zeta-d|^{-a}|\zeta|^{-1}d\zeta \\
&\leqslant C_{s_1}e^{-dF(x,y)}F(x,y)^{-a}|r-t|.
\end{aligned}$$

In the following estimate we use that $x \mapsto S'_>(x)S_>(s_0)^{-1}$ is norm continuous:

$$\begin{aligned}
&\left\|(S_{r>}(x)-\zeta)^{-1}S'_{r>}(x)(S_{r>}(x)-\zeta)^{-1}-(S_{t>}(x)-\zeta)^{-1}S'_{t>}(x)(S_{t>}(x)-\zeta)^{-1}\right\| \\
&\quad \leqslant \left\|\left[(S_{r>}(x)-\zeta)^{-1}-(S_{t>}(x)-\zeta)^{-1}\right]S'_{r>}(x)(S_{r>}(x)-\zeta)^{-1}\right\| \\
&\quad\quad + \left\|(S_{t>}(x)-\zeta)^{-1}\left[S'_{r>}(x)-S'_{t>}(x)\right](S_{r>}(x)-\zeta)^{-1}\right\| \\
&\quad\quad + \left\|(S_{t>}(x)-\zeta)^{-1}S'_{t>}(x)\left[(S_{r>}(x)-\zeta)^{-1}-(S_{t>}(x)-\zeta)^{-1}\right]\right\| \\
&\quad \overset{(3.22)}{\leqslant} C_{s_1}|\zeta|^{-1}|r-t|\left|\frac{\partial \xi_r(x)}{\partial x}\right|\left\|S'_>(\xi_r(x))S_>(s_0)^{-1}\right\|\left\|S_>(s_0)(S_{r>}(x)-\zeta)^{-1}\right\| \\
&\quad\quad + |\zeta|^{-1}\left[\left|\frac{\partial \xi_r(x)}{\partial x}-\frac{\partial \xi_t(x)}{\partial x}\right|\left\|S'_>(\xi_r(x))S_>(s_0)^{-1}\right\|\right. \\
&\quad\quad\quad \left.+\left|\frac{\partial \xi_t(x)}{\partial x}\right|\left\|\left[S'_>(\xi_r(x))-S'_>(\xi_t(x))\right]S_>(s_0)^{-1}\right\|\right]\left\|S_>(s_0)(S_{r>}(x)-\zeta)^{-1}\right\| \\
&\quad\quad + C_{s_1}\left\|S_>(s_0)(S_{t>}(x)-\zeta)^{-1}\right\|\left|\frac{\partial \xi_t(x)}{\partial x}\right|\left\|S'_>(\xi_t(x))S_>(s_0)^{-1}\right\||\zeta|^{-1}|r-t|
\end{aligned}$$

This yields

$$\Big\| \left(S_{r>}(x)-\zeta\right)^{-1} S'_{r>}(x) \left(S_{r>}(x)-\zeta\right)^{-1} - \left(S_{t>}(x)-\zeta\right)^{-1} S'_{t>}(x) \left(S_{t>}(x)-\zeta\right)^{-1} \Big\| \\ \leqslant C_\varepsilon |\zeta|^{-1}\varepsilon \qquad \text{for all } x\in[s_1,s_0] \text{ and } |r-t|\leqslant\delta. \tag{3.23}$$

In Lemma 3.3.1 it has been shown that D_r is an operator with horn singularity and thus, the properties of $W_>$ and $\widetilde{W}_>$, calculated in Lemma 3.1.2 on page 15, apply also to $W_{r>}$ and $\widetilde{W}_{r>}$, respectively.

$$\frac{\partial}{\partial x}\widetilde{W}_{r>}(x,y) = -x^{-\beta} S_{r>}(x)\widetilde{W}_{r>}(x,y) + \widetilde{R}_r(x,y)$$

with

$$\widetilde{R}_r(x,y) := -\frac{1}{2\pi i}\int_c e^{-\zeta F(x,y)} \left(S_{r>}(x)-\zeta\right)^{-1} S'_{r>}(x) \left(S_{r>}(x)-\zeta\right)^{-1} d\zeta.$$

Analogous to the proof of assertion 1 in the HO$^\infty$ case in Lemma 3.1.2 one shows

$$\left\|\widetilde{R}_r^j(x,y)\right\|_{\mathcal{L}(H)} \leqslant \underbrace{K \sup_{u\in[s_1,s_0]} \alpha(u)}_{=:\widetilde{K}_{s_1}} \left[C_{s_0} F(x,y)^{1-a}\right]^{j-1} e^{-dF(x,y)} F(x,y)^{-a}$$

for all $s_1 \leqslant y < x \leqslant s_0$ with

$$C_{s_0} = KB(1-a,1-a)C_4(s_0),$$

which does not depend on r (Actually, K depends on C_2 which is the same for all D_r). Next, we have to prove an analogous estimate for the difference $\widetilde{R}_r^j(x,y) - \widetilde{R}_t^j(x,y)$. We begin with estimating the difference:

$$\begin{aligned} \left\|\widetilde{R}_r(x,y) - \widetilde{R}_t(x,y)\right\| &= \Bigg\| \frac{1}{2\pi i}\int_c e^{-\zeta F(x,y)} \Big[-\left(S_{r>}(x)-\zeta\right)^{-1} S'_{r>}(x) \left(S_{r>}(x)-\zeta\right)^{-1} \\ &\qquad + \left(S_{t>}(x)-\zeta\right)^{-1} S'_{t>}(x) \left(S_{t>}(x)-\zeta\right)^{-1} \Big] d\zeta \Bigg\| \\ &\leqslant C e^{-dF(x,y)} F(x,y)^{-a} \int_c |\zeta - d|^{-a} \Big\| \left(S_{r>}(x)-\zeta\right)^{-1} S'_{r>}(x) \left(S_{r>}(x)-\zeta\right)^{-1} \\ &\qquad - \left(S_{t>}(x)-\zeta\right)^{-1} S'_{t>}(x) \left(S_{t>}(x)-\zeta\right)^{-1} \Big\| d\zeta \\ &\overset{(3.23)}{\leqslant} \widetilde{C}_\varepsilon e^{-dF(x,y)} F(x,y)^{-a}\varepsilon, \qquad \forall\ s_1\leqslant y<x\leqslant s_0 \text{ and } |r-t|\leqslant\delta. \end{aligned}$$

Next, we prove by induction over j that

$$\left\|\widetilde{R}_r^j(x,y) - \widetilde{R}_t^j(x,y)\right\| \leqslant \widetilde{C}_\varepsilon (C_\varepsilon F(x,y)^{1-a})^{j-1} e^{-dF(x,y)} F(x,y)^{-a}\varepsilon$$

for all $s_1 \leqslant y < x \leqslant s_0$ and $|r-t|\leqslant\delta$, where

$$C_\varepsilon := \max\left\{2\widetilde{K}_{s_1} s_0^\beta B(1-a,1-a), C_{s_0}\right\}.$$

The case $j = 1$ was treated above. Let us assume that we have proved the case j.

$$\begin{aligned}\left\|\widetilde{R}_r^{j+1}(x,y) - \widetilde{R}_t^{j+1}(x,y)\right\| &= \left\|\int_y^x \left[\widetilde{R}_r(x,z)\widetilde{R}_r^j(z,y) - \widetilde{R}_t(x,z)\widetilde{R}_t^j(z,y)\right]dz\right\| \\
&\leqslant \int_y^x \left[\left\|\widetilde{R}_r(x,z) - \widetilde{R}_t(x,z)\right\| \left\|\widetilde{R}_r^j(z,y)\right\| + \left\|\widetilde{R}_t(x,z)\right\| \left\|\widetilde{R}_r^j(z,y) - \widetilde{R}_t^j(z,y)\right\|\right]dz \\
&\leqslant \int_y^x \Big[\widetilde{C}_\varepsilon e^{-dF(x,z)}F(x,z)^{-a}\varepsilon\widetilde{K}_{s_1}\left[C_{s_0}F(z,y)^{1-a}\right]^{j-1} e^{-dF(z,y)}F(z,y)^{-a} \\
&\quad + \widetilde{K}_{s_1}e^{-dF(x,z)}F(x,z)^{-a}\widetilde{C}_\varepsilon\left[C_\varepsilon F(z,y)^{1-a}\right]^{j-1} e^{-dF(z,y)}F(z,y)^{-a}\varepsilon\Big]dz \\
&\leqslant \widetilde{C}_\varepsilon\widetilde{K}_{s_1}\left(C_{s_0}^{j-1} + C_\varepsilon^{j-1}\right)\left[F(x,y)^{1-a}\right]^{j-1} e^{-dF(x,y)}\varepsilon\int_y^x F(x,z)^{-a}F(z,y)^{-a}\frac{s_0^\beta}{z^\beta}dz \\
&\leqslant \widetilde{C}_\varepsilon\widetilde{K}_{s_1}2\left[C_\varepsilon F(x,y)^{1-a}\right]^{j-1} e^{-dF(x,y)}\varepsilon F(x,y)^{1-2a}s_0^\beta B(1-a,1-a) \\
&= \widetilde{C}_\varepsilon\left[C_\varepsilon F(x,y)^{1-a}\right]^j e^{-dF(x,y)}F(x,y)^{-a}\varepsilon \qquad \forall\ s_1 \leqslant y < x \leqslant s_0,\ |r-t| \leqslant \delta.\end{aligned}$$

Replacing the first $\widetilde{R}_r$ by $\widetilde{W}_{r>}$, gives for all $s_1 \leqslant y < x \leqslant s_0$ and $|r - t| \leqslant \delta$

$$\left\|\widetilde{W}_{r>}\tilde{*}\widetilde{R}_r^j(x,y) - \widetilde{W}_{t>}\tilde{*}\widetilde{R}_t^j(x,y)\right\| \leqslant \widetilde{C}_\varepsilon\left[C_\varepsilon F(x,y)^{1-a}\right]^j e^{-dF(x,y)}F(x,y)^{-a}\varepsilon.$$

Analogously to the calculations in the proof of Lemma 3.1.2 this implies

$$\|W_{r>}(x,y) - W_{t>}(x,y)\| \leqslant \frac{\widetilde{C}_\varepsilon}{1 - C_\varepsilon F(x,y)^{1-a}} e^{-dF(x,y)}F(x,y)^{-a}\varepsilon \tag{3.24}$$

for all $s_1 \leqslant y < x \leqslant s_0$ with $C_\varepsilon F(x,y)^{1-a} < 1$ and $|r - t| \leqslant \delta$.

Choose $A > 0$ small enough, such that $C_\varepsilon(2A)^{1-a} < 1$ and, such that there is a smallest natural number N_A with $s_0 < ({s_1}^{-\beta+1} - N_A(\beta-1)A)^{-\frac{1}{\beta-1}}$. This is satisfied if

$$A < \min\left\{\frac{1}{2}C_\varepsilon^{-\frac{1}{1-a}}, \frac{s_0^{-\beta+1}}{\beta-1}\right\}.$$

Let $y < x \in [s_1, s_0]$ and define

$$z_j := \left(y^{-\beta+1} - j(\beta-1)A\right)^{-\frac{1}{\beta-1}}, \quad j \in \left\{0, 1, \ldots, \left\lceil \frac{y^{-\beta+1}}{(\beta-1)A} - 1\right\rceil\right\}.$$

Then $z_0 = y$ and

$$F(z_j, z_{j-1}) = A, \quad \forall j \in \left\{1, \ldots, \left\lceil \frac{y^{-\beta+1}}{(\beta-1)A} - 1\right\rceil\right\}.$$

Let N be the natural number that satisfies $z_N \leqslant x < z_{N+1}$.

Case 1: $N \geqslant 1$

$$\begin{aligned}\|W_{r>}(x,y) - W_{t>}(x,y)\| \leqslant \|W_{r>}(x,z_{N-1}) - W_{t>}(x,z_{N-1})\| \prod_{j=1}^{N-1} \|W_{r>}(z_j,z_{j-1})\| \\ + \|W_{t>}(x,z_{N-1})\| \sum_{i=1}^{N-1} \|W_{t>}(z_{N-1},z_{N-2})\| \cdots \|W_{t>}(z_{N-i+1},z_{N-i})\| \cdot \\ \cdot \|W_{r>}(z_{N-i},z_{N-i-1}) - W_{t>}(z_{N-i},z_{N-i-1})\| \cdot \\ \cdot \|W_{r>}(z_{N-i-1},z_{N-i-2})\| \cdots \|W_{r>}(z_1,z_0)\| \\ \overset{(3.24)}{\leqslant} N\varepsilon \frac{\widetilde{C}_\varepsilon}{1 - C_\varepsilon F(x,z_{N-1})^{1-a}} e^{-dF(x,z_{N-1})} F(x,z_{N-1})^{-a} \\ \cdot \prod_{j=1}^{N-1} \frac{\widetilde{C}_\varepsilon}{1 - C_\varepsilon F(z_j,z_{j-1})^{1-a}} e^{-dF(z_j,z_{j-1})} F(z_j,z_{j-1})^{-a} \\ \leqslant N\varepsilon \frac{\widetilde{C}_\varepsilon}{1 - C_\varepsilon (2A)^{1-a}} e^{-dF(x,y)} A^{-a} \prod_{j=1}^{N-1} \frac{\widetilde{C}_\varepsilon}{1 - C_\varepsilon A^{1-a}} A^{-a} \\ \leqslant N \left(\frac{\widetilde{C}_\varepsilon A^{-a}}{1 - C_\varepsilon (2A)^{1-a}} \right)^N e^{-dF(x,y)} \varepsilon \\ \leqslant N_A \max \left\{ 1, \left(\frac{\widetilde{C}_\varepsilon A^{-a}}{1 - C_\varepsilon (2A)^{1-a}} \right)^{N_A} \right\} e^{-dF(x,y)} \varepsilon\end{aligned}$$

The last step is necessary to get a uniform constant with respect to x and y.

Case 2: $N = 0$, i.e. $y < x < z_1$

$$\begin{aligned}\|W_{r>}(x,y) - W_{t>}(x,y)\| &\leqslant \frac{\widetilde{C}_\varepsilon}{1 - C_\varepsilon F(x,y)^{1-a}} e^{-dF(x,y)} F(x,y)^{-a} \varepsilon \\ &\leqslant \frac{\widetilde{C}_\varepsilon}{1 - C_\varepsilon A^{1-a}} e^{-dF(x,y)} F(x,y)^{-a} \varepsilon\end{aligned}$$

Combining the two estimates, we get the result. By duality we find the same estimate for the expression where $>$ is replaced by $<$.

□

Lemma 3.3.6. [14] *If D is an operator with C^1-horn singularity, the map*

$$[0,1] \ni r \mapsto P_r \in \mathcal{L}(L^2(M,F), L^2(M,E))$$

is continuous.

[14] Lemma 4.1 in [Brü92] states a similar result for the cone case and gives a sketch of proof.

Proof. All members of the parametrix family $\{P_r\}_{r\in[0,1]}$ are equal on M_1. Therefore, we have to show that $r \mapsto P_r$ is continuous on $L^2(I, H_{\text{tot}})$. The idea of proof is to use the continuity property of $W_{r>}$ away from the singularity and the smallness of the parametrix close to the singularity. First, we calculate the estimates for the four terms that will appear in the full estimate.

Let $\varepsilon_0 > 0$, $\varphi \in C^\infty([0, s_0], [0, 1])$ with $\varphi|_{[0,s_1]} \equiv 0$ and $\varphi|_{[2s_1,s_0]} \equiv 1$, and $0 < a < \frac{1}{2}$:

$$\begin{aligned}
\|(P_{r>} - P_{t>})\varphi f\|^2_{L^2(I,Q_>H)} &= \int_{s_1}^{s_0} \left\| \int_{s_1}^{x} [W_{r>}(x,y) - W_{t>}(x,y)]\, \varphi(y) f(y) dy \right\|^2_{Q_>H} dx \\
&\leqslant \int_{s_1}^{s_0} \left(\int_{s_1}^{x} \|W_{r>}(x,y) - W_{t>}(x,y)\|_{\mathcal{L}(Q_>H)} \|f(y)\|_{Q_>H}\, dy \right)^2 dx \\
&\overset{CS}{\leqslant} \int_{s_1}^{s_0} \left(\int_{s_1}^{x} \|W_{r>}(x,y) - W_{t>}(x,y)\|^2_{\mathcal{L}(Q_>H)}\, dy \int_0^{s_0} \|f(y)\|^2_{Q_>H}\, dy \right) dx \\
&\overset{3.3.5}{\leqslant} C^2_{\varepsilon_0}\varepsilon_0^2 \int_{s_1}^{s_0} \int_{s_1}^{x} e^{-2dF(x,y)} \left(F(x,y)^{-a} + 1\right)^2 \frac{s_0^\beta}{y^\beta} dy dx\, \|f\|^2_{L^2(I,Q_>H)} \\
&\leqslant C^2_{\varepsilon_0} s_0^\beta \varepsilon_0^2 \int_0^{s_0} dx \int_0^\infty e^{-2dz} \left(z^{-a} + 1\right)^2 dz\, \|f\|^2_{L^2(I,Q_>H)} \\
&\leqslant C^2_{\varepsilon_0}\varepsilon_0^2\, \|f\|^2_{L^2(I,Q_>H)}\,.
\end{aligned}$$

$$\Rightarrow \quad \|(P_{r>} - P_{t>})\varphi\|_{\mathcal{L}(L^2(I,Q_>H))} \leqslant C_{\varepsilon_0}\varepsilon_0 \qquad \text{for all } |r-t| \leqslant \delta_0.$$

In the $<$-case we have to multiply by φ from the left:

$$\begin{aligned}
\|\varphi(P_{r<} - P_{t<})f\|^2_{L^2(I,Q_<H)} &= \int_{s_1}^{s_0} \left\| \varphi(x) \int_{x}^{s_0} [W_{r<}(x,y) - W_{t<}(x,y)]\, f(y) dy \right\|^2_{Q_<H} dx \\
&\overset{3.3.5}{\leqslant} C^2_{\varepsilon_0}\varepsilon_0^2 \int_{s_1}^{s_0} \int_{x}^{s_0} e^{-2dF(y,x)} \left(F(y,x)^{-a} + 1\right)^2 \frac{s_0^\beta}{y^\beta} dy dx\, \|f\|^2_{L^2(I,Q_<H)} \\
&\leqslant C^2_{\varepsilon_0} s_0^\beta \varepsilon_0^2 \int_{s_1}^{s_0} dx \int_0^\infty e^{-2dz} \left(z^{-a} + 1\right)^2 dz\, \|f\|^2_{L^2(I,Q_<H)}\,.
\end{aligned}$$

$$\Rightarrow \quad \|\varphi(P_{r<} - P_{t<})\|_{\mathcal{L}(L^2(I,Q_<H))} \leqslant C_{\varepsilon_0}\varepsilon_0 \qquad \text{for all } |r-t| \leqslant \delta_0.$$

Next, we calculate the two remaining terms which contain $1 - \varphi$.

$$\begin{aligned}
\|P_{r>}(1-\varphi)f\|^2_{L^2(I,Q_>H)} &= \int_0^{s_0} \left\| \int_0^{x} W_{r>}(x,y)(1-\varphi(y)) f(y) dy \right\|^2 dx \\
&\leqslant \int_0^{s_0} \left\| \int_0^{2s_1} W_{r>}(x,y) f(y) dy \right\|^2 dx \\
&\overset{3.1.2}{\leqslant} C^2 \int_0^{s_0} \left(\int_0^{2s_1} e^{-2dF(x,y)} (F(x,y)^{-a} + 1)^2 \frac{(2s_1)^\beta}{y^\beta} dy \right) dx\, \|f\|^2_{L^2(I,Q_>H)} \\
&= C^2 (2s_1)^\beta \int_0^{s_0} dx \left(\int_0^\infty e^{-2du} (u^{-a} + 1)^2 du \right) \|f\|^2_{L^2(I,Q_>H)}
\end{aligned}$$

$$\Rightarrow \quad \|P_{r>}(1-\varphi)f\|_{L^2(I,Q_>H)} \leqslant C {s_1}^{\frac{\beta}{2}}\, \|f\|_{L^2(I,Q_>H)}$$

$$\begin{aligned}\|(1-\varphi)P_{r<}f\|^2_{L^2(I,Q_<H)} &= \int_0^{s_0} \|(1-\varphi(x))(P_{r<}f)(x)\|^2\,dx \\ &\overset{3.1.6,\ 2.}{\leqslant} \int_0^{2s_1} x^\beta dx\, \|f\|^2_{L^2(I,Q_<H)} \\ \Rightarrow \quad \|(1-\varphi)P_{r<}f\|_{L^2(I,Q_<H)} &\leqslant C{s_1}^{\frac{\beta+1}{2}} \|f\|_{L^2(I,Q_<H)}\,,\end{aligned}$$

where the constants are independent of r since Lemma 3.1.2 depends only on C_2. Combining the four estimates above, we get for all $s_1 \in I$:

$$\begin{aligned}\|(P_r-P_t)f\|^2_{L^2(I,H_{\text{tot}})} &= \|(P_{r>}-P_{t>})(\varphi+1-\varphi)Q_>f\|^2_{L^2(I,Q_>H)} \\ &\quad + \underbrace{\left\|(\widetilde{P}-\widetilde{P})f\right\|^2_{L^2(I,\widetilde{H})}}_{=0} + \|(\varphi+1-\varphi)(P_{r<}-P_{t<})Q_<f\|^2_{L^2(I,Q_<H)} \\ &\leqslant 3C^2_{\varepsilon_0}\varepsilon_0^2\left(\|f\|^2_{L^2(I,Q_>H)} + \|f\|^2_{L^2(I,Q_<H)}\right) \\ &\quad + 6C^2{s_1}^\beta\left(\|f\|^2_{L^2(I,Q_>H)} + \|f\|^2_{L^2(I,Q_<H)}\right) \\ &\leqslant \left(3C^2_{\varepsilon_0}\varepsilon_0^2 + 6C^2{s_1}^\beta\right)\|f\|^2_{L^2(I,H_{\text{tot}})}\,.\end{aligned}$$

$$\Rightarrow \quad \|P_r-P_t\|_{\mathcal{L}(L^2(I,H_{\text{tot}}))} \leqslant \sqrt{3}C_{\varepsilon_0}\varepsilon_0 + \sqrt{6}C{s_1}^{\frac{\beta}{2}} \qquad \text{for all } |r-t| \leqslant \delta_0. \tag{3.25}$$

We proceed with the proof of continuity: Let $r \in [0,1]$ and $\varepsilon > 0$ be given. Choose φ and $s_1 > 0$, such that $\sqrt{6}C{s_1}^{\frac{\beta}{2}} \leqslant \frac{\varepsilon}{2}$. Then choose $\delta > 0$ small enough, such that $\sqrt{3}C_{\varepsilon_0}\varepsilon_0(\delta) \leqslant \frac{\varepsilon}{2}$ for all $|r-t| \leqslant \delta$. That gives

$$\|P_r-P_t\|^2_{\mathcal{L}(L^2(I,H_{\text{tot}}))} \overset{(3.25)}{\leqslant} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{for all } |r-t| \leqslant \delta.$$

□

3.4 Index formula

In the last step of the proof it is shown that the index of D and the index of the classical cone operator differ by a finite linear combination of residua of the eta function at $1, 2, \ldots, \dim N$. The index of the cone operator has been calculated in [BS88] and thus gives the index formula in our case.

Lemma 3.4.1. *Let $h \in C^\infty(I,(0,\infty))$ with $h(x) = x$ in a neighbourhood of 0 and $h(x) = x^\beta$ in a neighbourhood of s_0 and $\|h'\|_\infty > 0$. Let D^1 be an operator with horn singularity, such that*

$$D^1|_U \cong \left(\frac{\partial}{\partial x} + h(x)^{-1}S(s_0)\right) \oplus \left(\frac{\partial}{\partial x} + x^{-1}\widetilde{S}\right).$$

Then

$$\operatorname{ind} D^1_{\min} = \operatorname{ind} D^1_\delta - \sum_{-\frac{1}{2}<\lambda<0} \dim\ker(\widetilde{S}-\lambda),$$

where δ refers to the "Dirichlet" boundary conditions defined on page 664 in [BS88].

Proof. Choose $\beta_1 = 1$, $\beta_2 = \beta$, $h_1(x) = h(x)$ and $h_2(x) = x^\beta$. By Lemma 5.2 on page 673-674 in [LP98] it exists a homotopy h_β between these two functions, such that Theorem 4.8 on page 672 in [LP98] can be applied. It states $\operatorname{ind} D^1_{\min} = \operatorname{ind}(D_1)_{\min}$. The last equality follows from Formula (3.16) on page 674 in [BS88]. □

The following lemma translates the formulas for the resolvent kernel proved in [BS88] into formulas for the trace of the heat kernel. This is necessary since both kernels are applied in the proof of the index theorem.

Lemma 3.4.2. *Let*

$$T_{cone} := \frac{\partial}{\partial x} + \frac{1}{x} S_0$$

be the classical cone operator satisfying δ-boundary conditions and $\varphi \in C_0^\infty(U, [0,1])$ with $\varphi \equiv 1$ in a neighbourhood of the singularity. Then

$$\operatorname{tr} \varphi \left(e^{-tT^*_{cone}T_{cone}} - e^{-tT_{cone}T^*_{cone}} \right) = -\frac{1}{2}(\eta(S_0) + \dim\ker S_0) + \sum_{k \geqslant 1} \alpha_k \operatorname{Res}_{2k} \eta_{S_0},$$

where η_{S_0} is the eta function and $\eta(S_0)$ the eta invariant of S_0, and the $\alpha_k \in \mathbb{R}$ do not depend on φ or S_0.

Proof. Define $\Delta^+ := T^*_{\text{cone}} T_{\text{cone}}$ and $\Delta^- := T_{\text{cone}} T^*_{\text{cone}}$. On page 685 in [BS88] we find the asymptotic expansion

$$\begin{aligned} \operatorname{tr}(\Delta^\pm + z^2)^{-m} \sim_{z\to\infty} & \sum_{j=1}^\infty \left(\int_M \varphi_i p_j^\pm \right) z^{-j} + \sum_{j=1}^\infty \int_0^\infty \varphi(x) \sigma_j^\pm(x)(xz)^{-j} dx \\ & + \sum_{j=1}^\infty z^{-j} \int_0^\infty \frac{1}{(j-1)!} \xi^{j-1} \frac{\partial^{j-1}\sigma_\pm}{\partial x^{j-1}}(0,\xi) d\xi \\ & + \sum_{j=1}^\infty z^{-j} (\log z) (\sigma_j^\pm)^{(j-1)}(0)/(j-1)! \end{aligned}$$

Since $\sigma_\pm(x,\xi) = x^{2m-1}\sigma_\pm(1,\xi)$, it follows that $\sigma_j^\pm(x) = x^{2m-1}\sigma_j(1)$. Thus, the expansion reduces to:

$$\begin{aligned} \operatorname{tr}(\Delta^\pm + z^2)^{-m} \sim_{z\to\infty} & \sum_{j=1}^\infty z^{-j} \left(\int_M \varphi_i p_j^\pm + \int_0^\infty \varphi(x)\sigma_j^\pm(x) x^{-j} dx \right) \\ & + z^{-2m} \int_0^\infty \xi^{2m-1} \sigma_\pm(1,\xi) d\xi + z^{-2m} (\log z) \sigma_{2m}^\pm(1). \end{aligned}$$

Comparing this formula with the asymptotic expansion on page 30 in [BL96]:

$$\operatorname{tr}(\Delta^\pm + z^2)^{-m} \sim_{z\to\infty} \sum_{j=1}^\infty \sum_{k=0}^{k(j)} A_{jk}^{r,\pm}(m) z^{\alpha_j - 2m} \log^k z,$$

we find $\alpha_j = -j + 2m$ for $j \in \mathbb{N}$, $k(j) = 0$ for $j \in \mathbb{N}\setminus\{2m\}$ and $k(2m) = 1$ in our case.

$$\begin{aligned}
A^{r,\pm}_{2m\,1}(m) &= \sigma^{\pm}_{2m}(1)\\
A^{r,\pm}_{2m\,0}(m) &= \int_M \varphi_i p^{\pm}_{2m} + \int_0^\infty \varphi(x)\sigma^{\pm}_{2m}(x)x^{-2m}dx + \int_0^\infty \xi^{2m-1}\sigma_{\pm}(1,\xi)d\xi\\
A^{r,\pm}_{j0}(m) &= \int_M \varphi_i p^{\pm}_j + \int_0^\infty \varphi(x)\sigma^{\pm}_j(x)x^{-j}dx \qquad j \in \mathbb{N}\setminus\{2m\}
\end{aligned}$$

In our case the formulas given in Lemma 2.2 in [BL96] simplify to

$$\begin{aligned}
A^{h,\pm}_{2m\,1} &= -A^{r,\pm}_{2m\,1}(m)\frac{(m-1)!}{2\Gamma(m)} = -\frac{1}{2}A^{r,\pm}_{2m\,1}(m) = -\frac{1}{2}\sigma^{\pm}_{2m}(1)\\
A^{h,\pm}_{2m\,0} &= \Gamma(m)\left[\frac{A^{r,\pm}_{2m\,0}}{\Gamma(m)} + A^{r,\pm}_{2m\,1}(m)\frac{1}{2}\frac{d}{d\alpha}(\Gamma(-\alpha))^{-1}|_{\alpha=-m}\right]\\
&= \int_M \varphi_i p^{\pm}_{2m} + \int_0^\infty \varphi(x)\sigma^{\pm}_{2m}(x)x^{-2m}dx + \int_0^\infty \xi^{2m-1}\sigma_{\pm}(1,\xi)d\xi\\
&\quad + \frac{1}{2}\sigma^{\pm}_{2m}(1)\left(-\gamma + \sum_{k=1}^{m}\frac{1}{k}\right)\\
A^{h,\pm}_{j0} &= A^{r,\pm}_{j0}\frac{\Gamma(m)}{\Gamma(\frac{j}{2})} = \left(\int_M \varphi_i p^{\pm}_j + \int_0^\infty \varphi(x)\sigma^{\pm}_j(x)x^{-j}dx\right)\frac{\Gamma(m)}{\Gamma(\frac{j}{2})} \qquad j \in \mathbb{N}\setminus\{2m\},
\end{aligned}$$

where γ is the *Euler-Macheroni constant.*

$$e^{-t\Delta^\pm} \sim_{t\to 0} \sum_{j=1}^{\infty}\left(A^{h,\pm}_{j1}t^{-\frac{\alpha_j}{2}}\log t + A^{h,\pm}_{j0}t^{-\frac{\alpha_j}{2}}\right) = \sum_{j=1}^{\infty}\left(A^{h,\pm}_{j1}t^{\frac{j}{2}-m}\log t + A^{h,\pm}_{j0}t^{\frac{j}{2}-m}\right)$$

Since

$$\operatorname{tr}[e^{-t\Delta^+} - e^{-t\Delta^-}] = \operatorname{ind} D_\delta,$$

the terms $A^{h,+}_{2m\,1}$ and $A^{h,-}_{2m\,1}$ must cancel since they give the only log terms. This implies that $\sigma^{+}_{2m}(1) - \sigma^{-}_{2m}(1) = 0$ and $\sigma^{+}_{2m}(x) - \sigma^{-}_{2m}(x) = 0$. The constant term corresponds to $j = 2m$:

$$\begin{aligned}
\operatorname{ind} D_\delta &= A^{h,+}_{2m\,0} - A^{h,-}_{2m\,0}\\
&= \int_M \varphi_i(p^{+}_{2m} - p^{-}_{2m}) + \int_0^\infty \frac{\xi^{2m-1}}{(2m-1)!}\left(\frac{\partial^{2m-1}\sigma_+}{\partial x^{2m-1}}(0,\xi) - \frac{\partial^{2m-1}\sigma_-}{\partial x^{2m-1}}(0,\xi)\right)d\xi,
\end{aligned}$$

which is the same term as in Formula (4.32) in [BS88]. This gives

$$\operatorname{tr}\left(\varphi e^{-t\Delta^+} - \varphi e^{-t\Delta^-}\right) = -\frac{1}{2}(\eta_{S_0} + \dim\ker S_0) + \sum_{k\geqslant 1}\alpha_k \operatorname{Res}_{2k}\eta_{S_0}$$

by Formula (4.42c) in [BS88]. At the top of page 692 in the same paper the asserted properties of α_k are explained. □

Theorem 3.4.3. [15] *Assume that D is an operator with C^1-horn singularity. There are constants $\widetilde{\alpha}_k \in \mathbb{R}$, such that*

$$\begin{aligned}\operatorname{ind} D_{\min} = \int_{M_1} \omega_D - \frac{1}{2}\left(\eta(S(s_0)) + \sum_{\lambda \geqslant 0} \dim\ker(\widetilde{S} - \lambda) - \sum_{\lambda < 0} \dim\ker(\widetilde{S} - \lambda)\right) \\ - \sum_{-\frac{1}{2} < \lambda < 0} \dim\ker(\widetilde{S} - \lambda) + \sum_{k=1}^{\infty} \widetilde{\alpha}_k \operatorname{Res}_k \eta_{S(s_0)},\end{aligned}$$

where $\eta_{S(s_0)}$ is the eta-function and $\eta(S(s_0))$ the eta-invariant of $S(s_0)$, and ω_D denotes the index form of D, i.e. $\omega_D(p)$ is the constant term in the asymptotic expansion of

$$\operatorname{tr}\left(\varphi e^{-tD^*D}(p,p) - \varphi e^{-tDD^*}(p,p)\right), \quad p \in M \text{ as } t \to 0,$$

and the integral stands for a certain regularization of the possibly divergent integral.

Proof. For $\chi \in C^\infty(M, [0,1])$ let $\beta(\chi)$ be the constant term in the asymptotic expansion of

$$\operatorname{tr} \chi \left(e^{-t(D^1_\delta)^* D^1_\delta} - e^{-tD^1_\delta (D^1_\delta)^*}\right).$$

Choose $0 < s_1 < s_2 < s_0$ in such a way that $h(x) = x$ for all $x \in (0, s_1]$ and $h(x) = x^\beta$ for all $x \in [s_2, s_0]$. Let $\chi_1 \in C_0^\infty([0,s_1), [0,1])$, such that $\chi_1 \equiv 1$ in a neighbourhood of 0. Let $\chi_2 \in C_0^\infty((0,s_0),[0,1])$, such that $\chi_1 + \chi_2 \equiv 1$ on $[0, s_2]$. Put $\chi_3 := 1 - \chi_1 - \chi_2$ and extend it to the rest manifold by putting it equal to one there. Then

$$\beta(\chi_3) = \int_M \chi_3 \omega_D.$$

$D^1_\delta|_{(0,s_1]\times N} = T_{\mathrm{cone},\delta}|_{(0,s_1]\times N}$ with $S_0 = S(s_0) \oplus \widetilde{S}$. Thus, Lemma 3.4.2 shows

$$\begin{aligned}\beta(\chi_1) &= -\frac{1}{2}(\eta(S(s_0) \oplus \widetilde{S}) + \dim\ker S(s_0) \oplus \widetilde{S}) + \sum_{k=1}^{\lfloor \frac{\dim N}{2} \rfloor} \alpha_k \operatorname{Res}_{2k} \eta_{S(s_0)\oplus\widetilde{S}} \\ &= -\frac{1}{2}\left(\eta(S(s_0)) + \sum_{\lambda \geqslant 0} \dim\ker(\widetilde{S} - \lambda) - \sum_{\lambda < 0} \dim\ker(\widetilde{S} - \lambda)\right) \\ &\quad + \sum_{k=1}^{\lfloor \frac{\dim N}{2} \rfloor} \alpha_k \operatorname{Res}_{2k} \eta_{S(s_0)},\end{aligned}$$

where the right side is independent of the choice on χ_1.

On the part of U corresponding to χ_2 the operator D^1 restricted to H looks like

$$\frac{\partial}{\partial x} + \frac{1}{h(x)} S(s_0) = \frac{\partial}{\partial x} + \frac{\varphi(x)}{x} S(s_0),$$

[15] The theorem is similar to Theorem 4.2 in [Brü92] and to Theorem 4.2 in [Brü90]. The proof is given in [Brü90] and carries over to our case with some adaptations.

where $\varphi(x) = 1$ near 0 and $\varphi(x) = x^{-\beta+1}$ near s_0. Thus, we can apply Theorem 4.4.4[16] which gives

$$\beta(\chi_2) = \sum_{n=0}^{\dim N} \left[\int_0^{s_0} \chi_2(x) g_{0n}(x) x^{-1} dx\right] \operatorname{Res}_n \eta_{S(s_0)} =: \sum_{n=1}^{\dim N} \beta_n \operatorname{Res}_n \eta_{S(s_0)}.$$

By Lemma 3.4.1 and summation we find

$$\operatorname{ind} D_{\min} = \operatorname{ind} D^1_\delta - \sum_{-\frac{1}{2}<\lambda<0} \dim\ker(\widetilde{S}-\lambda) = \beta(\chi_1) + \beta(\chi_2) + \beta(\chi_3) - \sum_{-\frac{1}{2}<\lambda<0} \dim\ker(\widetilde{S}-\lambda).$$

Letting $s_2 \to s_0$ and choosing a sequence of χ_2 accordingly, proves the assertion. □

Corollary 3.4.4. [17] *For an operator with C^1-horn singularity D*

$$\dim \left(\mathcal{D}(D_{\max}) \big/ \mathcal{D}(D_{\min}) \right) = \sum_{|\lambda|<\frac{1}{2}} \dim\ker(\widetilde{S}-\lambda).$$

Proof. $-D^t$ is an operator with C^1-horn singularity and

$$-D^t|_U \cong \left(\frac{\partial}{\partial x} - x^{-\beta} S(x) - S_1(x)\right) \oplus \left(\frac{\partial}{\partial x} - x^{-1}\widetilde{S} - \widetilde{S}_1(x)\right).$$

This implies

$$\begin{aligned}
&\dim \left(\mathcal{D}(D_{\max}) \big/ \mathcal{D}(D_{\min}) \right) = \operatorname{ind} D_{\max} - \operatorname{ind} D_{\min} = -\operatorname{ind}\left(-D^t_{\min}\right) - \operatorname{ind} D_{\min} \\
&= -\int_{M_1} \omega_{(-D^t)} + \frac{1}{2}\left(\eta(-S(s_0)) + \sum_{\lambda\geqslant 0} \dim\ker(-\widetilde{S}-\lambda) - \sum_{\lambda<0} \dim\ker(-\widetilde{S}-\lambda)\right) \\
&\quad + \sum_{-\frac{1}{2}<\lambda<0} \dim\ker(-\widetilde{S}-\lambda) - \sum_{k=1}^{\infty} \widetilde{\alpha}_k \operatorname{Res}_k \eta_{-S(s_0)} \\
&\quad - \int_{M_1} \omega_D + \frac{1}{2}\left(\eta(S(s_0)) + \sum_{\lambda\geqslant 0} \dim\ker(\widetilde{S}-\lambda) + \sum_{\lambda<0} \dim\ker(\widetilde{S}-\lambda)\right) \\
&\quad + \sum_{-\frac{1}{2}<\lambda<0} \dim\ker(\widetilde{S}-\lambda) - \sum_{k=1}^{\infty} \widetilde{\alpha}_k \operatorname{Res}_k \eta_{S(s_0)} \\
&= \left(\frac{1}{2}\sum_{\lambda\geqslant 0} - \frac{1}{2}\sum_{\lambda<0} + \sum_{-\frac{1}{2}<\lambda<0}\right)\left[\dim\ker(\widetilde{S}+\lambda) + \dim\ker(\widetilde{S}-\lambda)\right] = \sum_{|\lambda|<\frac{1}{2}} \dim\ker(\widetilde{S}-\lambda).
\end{aligned}$$

□

[16] This theorem is an explicit version of Theorem 4.1 in [Brü88].
[17] This is similar to Corollary 4.3 in [Brü90].

4 Methods

4.1 An abstract index theorem for a one-parameter family of closed operators with variable domain

Continuous families of bounded Fredholm operators have a constant index. For families of unbounded operators one can define a certain kind of "continuity", such that there is a similar result in this case. This result goes back to [CL63], but the version proved here was stated implicitly in [Brü92] and explicitly in [LP98].

Lemma 4.1.1. [1] *Let $D_r : \mathcal{H}_1 \supset \mathcal{D}(D_r) \to \mathcal{H}_2$ be a one-parameter family of densely defined closed Fredholm operators of Hilbert spaces. We define the closed operator*

$$E_r := \begin{pmatrix} \mathbb{1} & -D_r^* \\ D_r & \mathbb{1} \end{pmatrix} : \mathcal{D}(D_r) \oplus \mathcal{D}(D_r^*) \to \mathcal{H}_1 \oplus \mathcal{H}_2.$$

Then E_r is a bijective closed operator and E_r^{-1} is bounded. If the map

$$[r_1, r_2] \ni r \mapsto E_r^{-1} \in \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2)$$

is continuous, $\operatorname{ind} D_r$ *is independent of r.*

The problem with proving this lemma is that D_r is a family of closed operators and therefore, we cannot apply the usual theorem. We have to define convergence for sequences of closed operators. This can be accomplished by defining a so-called *gap function* on the graphs of the operators.

Definition 4.1.2. Let S, $T : \mathcal{H}_1 \to \mathcal{H}_2$ be closed operators and $\mathcal{G}(T)$ the graph of T. Then we define

$$\begin{aligned} \operatorname{dist}((x, Sx), \mathcal{G}(T)) &:= \inf_{y \in \mathcal{D}(T)} \|(x, Sx) - (y, Ty)\| \qquad x \in \mathcal{D}(S) \\ \delta(S, T) &:= \sup_{x \in \mathcal{D}(S),\ \|x\|_{\mathcal{G}(S)} = 1} \operatorname{dist}((x, Sx), \mathcal{G}(T)) \end{aligned}$$

and finally the *gap function*

$$\widehat{\delta}(S, T) := \max\{\delta(S, T), \delta(T, S)\}.$$

We say that a sequence of closed operators $S_n : \mathcal{H}_1 \to \mathcal{H}_2$, $n \in \mathbb{N}$ *converges to S in the generalized sense* if and only if

$$\widehat{\delta}(S_n, S) \overset{n\to\infty}{\longrightarrow} 0.$$

[1] The statement of the lemma appeared first in the proof of Theorem 4.1 in [BS88]. Later on, it was formulated and proved as Lemma 4.5 in [LP98]. This lemma and its proof are restated here for the sake of completeness.

The following properties of the gap function were originally proved in [CL63], but we give the references in the well-known book [Kat95].

Lemma 4.1.3. *1. If S and T are densely defined closed operators, then $\widehat{\delta}(S^*, T^*) = \widehat{\delta}(S, T)$. ([Kat95] Theorem IV.2.18, page 204)*

2. If S and T are invertible closed operators, then $\widehat{\delta}(S^{-1}, T^{-1}) = \widehat{\delta}(S, T)$. ([Kat95] Theorem IV.2.20, page 205)

3. For bounded operators convergence in the generalized sense implies convergence in norm and vice versa. ([Kat95] Remark IV.2.16, page 204)

4. Let S be a closed Fredholm operator. It exists a constant δ, such that all closed operators T with $\widehat{\delta}(S, T) \leqslant \delta$ are Fredholm and $\operatorname{ind} S = \operatorname{ind} T$. In particular, that implies, if $r \mapsto T_r$ is a $\widehat{\delta}$-continuous function from an interval into the closed Fredholm operators, $\operatorname{ind} T_r$ is constant. ([Kat95] Remark IV.5.17, page 235, the footnote there gives further references)

Proof of Lemma 4.1.1.

- E_r is bijective: In order to prove that property, we have to understand the following isomorphisms:

$$\begin{aligned} V &: \mathcal{H}_2 \oplus \mathcal{H}_1 \longrightarrow \mathcal{H}_1 \oplus \mathcal{H}_2, & V(f, g) &:= (-g, f) \\ \iota_1 &: \mathcal{D}(D_r) \longrightarrow \mathcal{G}(D_r), & \iota_1(f) &:= (f, D_r f) \\ \iota_2 &: \mathcal{D}(D_r^*) \longrightarrow \mathcal{G}(D_r^*), & \iota_2(g) &:= (g, D_r^* g). \end{aligned}$$

  We apply the formula $V\mathcal{G}(D_r^*) = \mathcal{G}(D_r)^\perp$ ([Kat95], page 168) and find

$$\begin{array}{ccc} \mathcal{D}(D_r) \oplus \mathcal{D}(D_r^*) & \overset{\iota_1 \oplus V \circ \iota_2}{\longrightarrow} & \mathcal{G}(D_r) \oplus \overbrace{V\mathcal{G}(D_r^*)}^{=\mathcal{G}(D_r)^\perp} = \mathcal{H}_1 \oplus \mathcal{H}_2 \\ (f, g) & \longmapsto & (f, D_r f) + (-D_r^* g, g) = (f - D_r^* g, D_r f + g) = E_r(f, g). \end{array}$$

  Thus, E_r is a composition of isomorphisms and therefore bijective.

- E_r is closed: Let $(f_n, g_n)_{n\in\mathbb{N}} \in \mathcal{D}(D_r) \oplus \mathcal{D}(D_r^*)$, such that $(f_n, g_n) \to (f, g)$ and $E_r(f_n, g_n) \to (\widehat{f}, \widehat{g}) \in \mathcal{H}_1 \oplus \mathcal{H}_2$.

$$\begin{array}{rcl} f_n - D_r^* g_n \overset{n\to\infty}{\longrightarrow} \widehat{f} & \overset{f_n \overset{n\to\infty}{\longrightarrow} f}{\Longrightarrow} & D_r^* g_n \overset{n\to\infty}{\longrightarrow} f - \widehat{f} \\ & \overset{D_r^* \text{ closed}}{\Longrightarrow} & g \in \mathcal{D}(D_r^*) \text{ and } D_r^* g = f - \widehat{f} \\ D_r f_n + g_n \overset{n\to\infty}{\longrightarrow} \widehat{g} & \overset{g_n \overset{n\to\infty}{\longrightarrow} g}{\Longrightarrow} & D_r f_n \overset{n\to\infty}{\longrightarrow} \widehat{g} - g \\ & \overset{D_r \text{ closed}}{\Longrightarrow} & f \in \mathcal{D}(D_r) \text{ and } D_r f = \widehat{g} - g \end{array}$$

  Thus, $(f, g) \in \mathcal{D}(D_r) \oplus \mathcal{D}(D_r^*)$ and

$$E_r(f, g) = (f - D_r^* g, D_r f + g) = \Big(f - \Big(f - \widehat{f}\,\Big), (\widehat{g} - g) + g\Big) = \Big(\widehat{f}\,, \widehat{g}\Big).$$

- $E_r^{-1} : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{D}(D_r) \oplus \mathcal{D}(D_r^*)$ is closed and due to the closed graph theorem, it is bounded which proves the first assertion of the lemma.

- The norms $\|\cdot\|_{\mathcal{G}(E_r)}$ and $(\|\cdot\|^2_{\mathcal{G}(D_r)} + \|\cdot\|^2_{\mathcal{G}(D_r^*)})^{\frac{1}{2}}$ are equivalent on $\mathcal{D}(D_r) \oplus \mathcal{D}(D_r^*)$: Let $f \in \mathcal{D}(D_r),\ g \in \mathcal{D}(D_r^*)$.
$$\begin{aligned}\|(f,g)\|^2_{\mathcal{G}(E_r)} &= \|(f,g)\|^2 + \|E_r(f,g)\|^2 = \|f\|^2 + \|g\|^2 + \|f - D_r^* g\|^2 + \|D_r f + g\|^2 \\ &\leqslant 3\,\|f\|^2 + 3\,\|g\|^2 + 2\,\|D_r^* g\|^2 + 2\,\|D_r f\|^2 \leqslant 3\left(\|f\|^2_{\mathcal{G}(D_r)} + \|g\|^2_{\mathcal{G}(D_r^*)}\right)\end{aligned}$$
$$\begin{aligned}\|f\|^2_{\mathcal{G}(D_r)} + \|g\|^2_{\mathcal{G}(D_r^*)} &= \|f\|^2 + \|D_r f\|^2 + \|g\|^2 + \|D_r^* g\|^2 \\ &= \|f\|^2 + \|D_r f + g - g\|^2 + \|g\|^2 + \|f - (f - D_r^* g)\|^2 \\ &\leqslant 3\,\|f\|^2 + 3\,\|g\|^2 + 2\,\|f - D_r^* g\|^2 + 2\,\|D_r f + g\|^2 \leqslant 3\,\|(f,g)\|^2_{\mathcal{G}(E_r)}\end{aligned}$$
- Finally, everything is put together: If the map $r \mapsto E_r^{-1}$ is continuous in the norm topology, it is also continuous in the generalized topology (4.1.3, 3). This implies that the map $r \mapsto E_r$ is continuous in the generalized sense (4.1.3, 2). Due to the equivalence of norms, this implies the continuity of $r \mapsto D_r$ and $r \mapsto (D_r)^*$ in the generalized sense. This implies $\operatorname{ind} D_r$ is constant by Lemma 4.1.3, 4.

□

4.2 Evolution equations

The scope of this section is to give a self-sufficient presentation of the solution theory of the equation
$$\frac{\partial}{\partial t} + A(t) = 0,$$
where $A(t)$ is a family of strongly continuously differentiable operators with common dense domain $\mathcal{D}(A) \subset E$. The statements and proofs have been taken from [Kre71]. Unfortunately, Theorem II.2.1 in [Kre71] cannot be applied directly. Therefore, we state and prove a slightly different version (Theorem 4.2.6).

We fix the following notation for this section: $T \in (0,\infty)$ and
$$\triangle := \left\{(t,s) \in \mathbb{R}^2 \mid 0 \leqslant s \leqslant t \leqslant T\right\}.$$

Definition 4.2.1. Let $A(t),\ t \in [0,T]$ be a family of operators with a common domain $\mathcal{D}(A)$ that is dense in E. We say that

- A is called *strongly continuous* if the functions $t \mapsto A(t)x$ are continuous for all $x \in \mathcal{D}(A)$.
- A is called *strongly differentiable* if the functions $t \mapsto A(t)x$ are differentiable for all $x \in \mathcal{D}(A)$.
- A is called *strongly continuously differentiable* if the functions $t \mapsto A(t)x$ are continuously differentiable for all $x \in \mathcal{D}(A)$.
- Let $A_n(t)$ be a sequence of families of unbounded operators sharing the common domain $\mathcal{D}(A)$. $A_n(t)$ converges strongly to $A(t)$ if and only if
$$A_n(t)x \overset{n\to\infty}{\longrightarrow} A(t)x, \qquad \forall\, x \in \mathcal{D}(A).$$

Remark: A strongly continuous family $[0,T] \ni t \mapsto A(t) \in \mathcal{L}(E)$ is uniformly bounded: Since $t \mapsto A(t)$ is strongly continuous and the interval $[0,T]$ is compact, it follows

$$\sup_{t\in[0,T]} \|A(t)x\| < \infty, \quad \forall x \in E.$$

Since $A(t)$ is bounded for all $t \in [0,T]$, the theorem of Banach-Steinhaus implies

$$\sup_{t\in[0,T]} \|A(t)\| < \infty.$$

We will use the following property of strongly continuous families of *bounded* operators.

Lemma 4.2.2. *Let E be a Banach-space and F a normed vector space.*

1. *If the family $[a,b] \ni t \mapsto A(t) \in \mathcal{L}(E,F)$ is strongly continuously differentiable, then it is Lipschitz continuous in norm.*[2]

2. *If the family $[a,b] \ni t \mapsto A(t) \in \mathcal{L}(E,F)$ is strongly differentiable and A' is continuous in norm, then A is continuously differentiable in norm.*

3. *If the family $[a,b] \ni t \mapsto A(t) \in \mathcal{L}(E,F)$ is two-times strongly continuously differentiable, then it is also continuously differentiable in norm.*

Proof. Ad 1. For x in E the family $t \mapsto A'(t)x$ is continuous and $[a,b]$ is compact. This implies

$$\sup_{a\leqslant s<t\leqslant b} \left\| \frac{A(t)-A(s)}{t-s} x \right\| = \sup_{a\leqslant s<t\leqslant b} \left\| \frac{1}{t-s} \int_s^t A'(z)x dz \right\| \leqslant \sup_{a\leqslant z\leqslant b} \left\| A'(z)x \right\| < \infty.$$

Since $(t-s)^{-1}(A(t)-A(s))$ are bounded operators for all $a \leqslant s < t \leqslant b$, it follows by the Theorem of Banach-Steinhaus

$$C := \sup_{a\leqslant s<t\leqslant b} \left\| \frac{A(t)-A(s)}{t-s} \right\| < \infty$$

and

$$\|A(t)-A(s)\| \leqslant C|t-s| \quad \text{ for all } t,s \in [a,b].$$

Ad 2. Let $t \in [a,b]$ and $\varepsilon > 0$. Since A' is continuous in norm, there exists a $\delta = \delta(\varepsilon,t)$, such that

$$\left\| A'(t) - A'(s) \right\| \leqslant \varepsilon \text{ for all } s \in [a,b] \text{ with } |s-t| \leqslant \delta.$$

Let $x \in E$. This implies

$$\begin{aligned} \left\| \left[\frac{A(t+h)-A(t)}{h} - A'(t) \right] x \right\| &= \left\| \frac{1}{h} \int_t^{t+h} A'(z) - A'(t) dz \; x \right\| \\ &\leqslant \frac{1}{|h|} \left| \int_t^{t+h} \left\| A'(z) - A'(t) \right\| dz \right| \|x\| \leqslant \varepsilon \|x\| \end{aligned}$$

[2]This is Lemma 3.5 on page 11 in [Kre71].

and

$$\Rightarrow \qquad \left\| \frac{A(t+h)-A(t)}{h} - A'(t) \right\| \leqslant \varepsilon \text{ for all } h \neq 0 \text{ with } |h| \leqslant \delta.$$

Ad 3. Since by assumption A' is strongly continuously differentiable, 1. implies that A' is continuous in norm. Since A is also strongly differentiable, the assertion follows from 2. □

Lemma 4.2.3. [3] *Let $A(t) \in \mathcal{L}(E)$ be a strongly continuous family of bounded operators for all $t \in [0,T]$. Then there exists a unique* evolution operator $U : \triangle \to \mathcal{L}(E)$, *i.e.*

- *U is strongly differentiable in t and s with*

$$\frac{\partial}{\partial t} U(t,s) = A(t)U(t,s) \quad \textit{and} \quad \frac{\partial}{\partial s} U(t,s) = -U(t,s)A(s).$$

- $U(t,\tau)U(\tau,s) = U(t,s)$ *and* $U(s,s) = \mathbb{1}, \quad \forall\ 0 \leqslant t \leqslant \tau \leqslant s \leqslant T.$

Proof. The solution operator can be constructed by applying the method of successive approximation to the Volterra integral equation

$$U(t,s) = \mathbb{1} + \int_s^t A(\tau)U(\tau,s)d\tau.$$

Define $P_0(t,s) := \mathbb{1}$ and for $n \in \mathbb{N}$

$$P_n(t,s) := \int_s^t A(\tau_1) \int_s^{\tau_1} A(\tau_2) \cdots \int_s^{\tau_{n-1}} A(\tau_n) d\tau_n \cdots d\tau_1 = \int_s^t A(\tau) P_{n-1}(\tau,s) d\tau.$$

- P_n is uniformly bounded:

$$\begin{aligned} \|P_n(t,s)\| &\leqslant \left(\sup_{\tau \in [0,T]} \|A(\tau)\| \right)^n \int_0^T \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} d\tau_n \cdots d\tau_1 \\ &= \left(\sup_{\tau \in [0,T]} \|A(\tau)\| \right)^n \frac{T^n}{n!} =: \frac{(CT)^n}{n!}. \end{aligned}$$

- P_n is uniformly continuous in t:

$$\begin{aligned} \|P_n(t_1,s) - P_n(t_2,s)\| &= \left\| \int_s^{t_1} A(\tau)P_{n-1}(\tau,s)d\tau - \int_s^{t_2} A(\tau)P_{n-1}(\tau,s)d\tau \right\| \\ &= \left\| \int_{t_1}^{t_2} A(\tau)P_{n-1}(\tau,s)d\tau \right\| \leqslant C \frac{(CT)^{n-1}}{(n-1)!} |t_2 - t_1|. \end{aligned}$$

[3] This is explained on page 188f. in [Kre71]. We formulated the lemma and carried out the details of the proof.

- P_n is strongly differentiable in t:

$$\begin{aligned}
&\left\| \frac{1}{h}[P_n(t+h,s)x - P_n(t,s)x] - A(t)P_{n-1}(t,s)x \right\| \\
&\quad = \left\| \frac{1}{h}\left[\int_s^{t+h} A(\tau)P_{n-1}(\tau,s)d\tau x - \int_s^t A(\tau)P_{n-1}(\tau)d\tau x\right] - A(t)P_{n-1}(t,s)x \right\| \\
&\quad = \left\| \frac{1}{h}\int_t^{t+h} A(\tau)P_{n-1}(\tau,s)d\tau x - A(t)P_{n-1}(t,s)x \right\| \\
&\quad \leqslant \frac{1}{|h|}\left| \int_t^{t+h} \|[A(\tau)-A(t)]\, P_{n-1}(\tau,s)x\| + \|A(t)\|\, \|P_{n-1}(\tau,s) - P_{n-1}(t,s)\|\, \|x\|\, d\tau \right|
\end{aligned}$$

The first term on the right becomes arbitrary small for $h \to 0$ since A is strongly uniformly continuous. The second term becomes small since P_n is uniformly norm continuous in t. Thus, P_n is strongly differentiable in t with derivative

$$\frac{\partial}{\partial t}P_n(t,s) = A(t)P_{n-1}(t,s).$$

Define

$$U_n(t,s) := \sum_{k=0}^{n} P_k(t,s).$$

- $\|U_n(t,s)\| \leqslant \sum_{k=0}^{n} \frac{(CT)^k}{k!} \leqslant e^{CT}$
- U_n is strongly differentiable in t:

$$\begin{aligned}
&\frac{\partial}{\partial t}U_0(t,s) = \frac{\partial}{\partial t}P_0(t,s) = \frac{\partial}{\partial t}\mathbb{1} = 0 \qquad \text{and for } n \in \mathbb{N} \\
&\frac{\partial}{\partial t}U_n(t,s) = \sum_{k=0}^{n}\frac{\partial}{\partial t}P_k(t,s) = \sum_{k=1}^{n} A(t)P_{k-1}(t,s) = A(t)\sum_{k=0}^{n-1}P_k(t,s) = A(t)U_{n-1}(t,s).
\end{aligned}$$

- U_n converges uniformly to a uniformly bounded operator U: Let $n > m \geqslant N \in \mathbb{N}$

$$\|U_n(t,s) - U_m(t,s)\| \leqslant \sum_{k=m+1}^{n} \|P_k(t,s)\| \leqslant \sum_{k=m+1}^{n} \frac{(CT)^k}{k!} \leqslant \frac{(CT)^N}{N!}e^{CT}.$$

Since E is Banach and thus $\mathcal{L}(E)$ is Banach, the last equation implies that U_n converges uniformly to a bounded operator U. The bound is uniform:

$$\|U(t,s)x\| \leqslant \underbrace{\|[U(t,s) - U_n(t,s)]\, x\|}_{\overset{n\to\infty}{\longrightarrow}0} + \underbrace{\|U_n(t,s)x\|}_{\leqslant e^{CT}\|x\|} \quad \forall n \in \mathbb{N}, \quad \Rightarrow \|U(t,s)\| \leqslant e^{CT}.$$

- $\frac{\partial}{\partial t}U_n(t,s)$ converges uniformly to $A(t)U(t,s)$:

$$\begin{aligned}
\left\| \frac{\partial}{\partial t}U_n(t,s) - A(t)U(t,s) \right\| &= \|A(t)U_{n-1}(t,s) - A(t)U(t,s)\| \\
&\leqslant \|A(t)\|\, \|U_{n-1}(t,s) - U(t,s)\| \leqslant C\, \|U_{n-1}(t,s) - U(t,s)\|\,.
\end{aligned}$$

- U is strongly differentiable in t and
$$\frac{\partial}{\partial t}U(t,s) = \frac{\partial}{\partial t}\lim_{n\to\infty}U_n(t,s) = \lim_{n\to\infty}\frac{\partial}{\partial t}U_n(t,s) = \lim_{n\to\infty}A(t)U_{n-1}(t,s) = A(t)U(t,s).$$

Next, we look at the reverse Volterra equation
$$V(t,s) = \mathbb{1} + \int_s^t V(t,\tau)A(\tau)d\tau.$$
We define $Q_0(t,s) := \mathbb{1}$ and $Q_n(t,s) := \int_s^t Q_{n-1}(t,\tau)A(\tau)d\tau,\ \forall n \in \mathbb{N}$ and find the properties:

- Q_n is uniformly bounded: $\|Q_n(t,s)\| \leqslant \frac{(CT)^n}{n!}$
- Q_n is uniformly continuous in s: $\|Q_n(t,s_2) - Q_n(t,s_1)\| \leqslant C\frac{(CT)^{n-1}}{(n-1)!}|s_2 - s_1|$
- Q_n is strongly differentiable in s:
$$\begin{aligned}
&\left\|\frac{1}{h}[Q_n(t,s+h) - Q_n(t,s)]x + Q_{n-1}(t,s)A(s)x\right\| \\
&\quad = \left\|\frac{1}{h}\left[\int_{s+h}^t Q_{n-1}(t,\tau)A(\tau)d\tau - \int_s^t Q_{n-1}(t,\tau)A(\tau)d\tau\right]x + Q_{n-1}(t,s)A(s)x\right\| \\
&\quad = \left\|-\frac{1}{h}\int_s^{s+h} Q_{n-1}(t,\tau)A(\tau)d\tau x + Q_{n-1}(t,s)A(s)x\right\| \\
&\quad \leqslant \frac{1}{|h|}\left|\int_s^{s+h}\|Q_{n-1}(t,\tau) - Q_{n-1}(t,s)\|\,\|A(\tau)x\| + \|Q_{n-1}(t,s)\|\,\|[A(\tau) - A(s)]\,x\|\,d\tau\right|
\end{aligned}$$
The first term on the right becomes arbitrary small since Q_n is uniformly norm continuous in s. The second term becomes small since A is strongly uniformly continuous. Thus, Q_n is strongly differentiable in s with derivative
$$\frac{\partial}{\partial s}Q_n(t,s) := -Q_{n-1}(t,s)A(s).$$

Define
$$V_n(t,s) := \sum_{k=0}^n Q_k(t,s).$$
The properties of V_n are proved in a similar way as the properties of U_n above: V_n converges uniformly to a bounded operator V that is strongly differentiable in s with derivative
$$\frac{\partial}{\partial s}V(t,s) = -V(t,s)A(s) \quad \text{and} \quad V(t,t) = \mathbb{1}.$$
Let $0 \leqslant t \leqslant s \leqslant \tau \leqslant T$:
$$\begin{aligned}
&\frac{\partial}{\partial s}[V(t,s)U(s,\tau)] = -V(t,s)A(s)U(s,\tau) + V(t,s)A(s)U(s,t) = 0 \\
\Rightarrow\quad & U(t,\tau) = V(t,t)U(t,\tau) = V(t,s)U(s,\tau) = V(t,\tau)U(\tau,\tau) = V(t,\tau) \\
\Rightarrow\quad & U(t,s)U(s,\tau) = U(t,\tau).
\end{aligned}$$
Furthermore, these equations imply that U is uniquely given and strongly differentiable in t and s with the asserted partial derivatives. □

Theorem 4.2.4. [4] *Let $(A_n(t))_{n\in\mathbb{N}}$ be a sequence of strongly continuous families of bounded operators on $[0,T]$, and let $U_n(t,s)$ be the sequence of evolution operators corresponding to them. Suppose that $B(t)$ is a strongly continuous family of bounded operators on $[0,T]$. Let $\widetilde{U}_n(t,s)$ be the evolution operators corresponding to the strongly continuous families of bounded operators $\widetilde{A}_n(t) = A_n(t) + B(t)$.*
If the sequence $U_n(t,s)$ is uniformly bounded with respect to $(t,s) \in \triangle$ and $n \in \mathbb{N}$, then the sequence $\widetilde{U}_n(t,s)$ is uniformly bounded with respect to $(t,s) \in \triangle$ and $n \in \mathbb{N}$.
If the sequence $U_n(t,s)$ converges strongly and uniformly with respect to $(t,s) \in \triangle$, then the sequence $\widetilde{U}_n(t,s)$ also converges strongly and uniformly with respect to $(t,s) \in \triangle$.

Proof. First, we compute an equality that connects U and $\widetilde{U}$. Let $(t,s) \in \triangle$.

$$\begin{aligned}
\widetilde{U}_n(t,s) &= U_n(t,s) + U_n(t,t)\widetilde{U}_n(t,s) - U_n(t,s)\widetilde{U}_n(s,s) \\
&= U_n(t,s) + \int_s^t \frac{\partial}{\partial \tau}\left[U_n(t,\tau)\widetilde{U}_n(\tau,s)\right] d\tau \\
&= U_n(t,s) + \int_s^t \left[-U_n(t,\tau)A_n(\tau)\widetilde{U}_n(\tau,s) + U_n(t,\tau)\left[A_n(\tau)+B(\tau)\right]\widetilde{U}_n(\tau,s)\right] d\tau
\end{aligned}$$

$$\Rightarrow \qquad \widetilde{U}_n(t,s) = U_n(t,s) + \int_s^t U_n(t,\tau)B(\tau)\widetilde{U}_n(\tau,s)d\tau \tag{4.1}$$

If $U_n(t,s)$ is uniformly bounded with respect to $(t,s) \in \Delta$ and $n \in \mathbb{N}$, we define

$$M := \sup_{(t,s)\in\triangle, n\in\mathbb{N}} \|U_n(t,s)\|.$$

With $C := \sup_{\tau\in[0,T]} \|B(\tau)\|$, Formula (4.1) implies

$$\left\|\widetilde{U}_n(t,s)\right\| \leqslant M + MC\int_s^t \left\|\widetilde{U}_n(\tau,s)\right\| d\tau, \quad \forall\ (t,s) \in \triangle \text{ and } n \in \mathbb{N}.$$

Applying Gronwall's inequality, gives

$$\left\|\widetilde{U}_n(t,s)\right\| \leqslant Me^{MCT}, \quad \forall\ (t,s) \in \triangle \text{ and } n \in \mathbb{N},$$

i.e. $\widetilde{U}_n(t,s)$ is uniformly bounded.

We proceed with the second assertion. Let

$$G(E) := \left(\left\{\widehat{x} = (x_n)_{n\in\mathbb{N}} \subset E \;\middle|\; \exists\ x \in E : \lim_{n\to\infty} x_n = x\right\}, \|\widehat{x}\| = \sup_{n\in\mathbb{N}} \|x_n\|\right)$$

be the Banach space of convergent sequences in E. Define the operator

$$\widehat{U}(t,s)\widehat{x} := (U_n(t,s)x_n)_{n\in\mathbb{N}}.$$

[4] This is Theorem 2.1 in [Kre71], page 190. The proof is also taken from there and given here for the sake of completeness. Furthermore, we have carried out the construction of the solution operator of the Volterra integral equation that has been omitted.

Since $x_n \overset{n\to\infty}{\longrightarrow} x$ and $U_n(t,s)$ converges strongly and uniformly with respect to $(t,s) \in \triangle$ and is uniformly bounded, we find

$$\|U_n(t,s)x_n - U(t,s)x\| \leqslant \underbrace{\|U_n(t,s)\|}_{\leqslant M} \|x_n - x\| + \|[U_n(t,s) - U(t,s)]\, x\| \overset{n\to\infty}{\longrightarrow} 0$$

uniformly with respect to $(t,s) \in \triangle$.

$$\left\|\widehat{U}(t,s)\widehat{x}\right\| = \sup_{n\in\mathbb{N}} \|U_n(t,s)x_n\| \leqslant \sup_{n\in\mathbb{N}} \|U_n(t,s)\| \, \|x_n\| \leqslant M \sup_{n\in\mathbb{N}} \|x_n\| = M \, \|\widehat{x}\|$$

Thus, $\widehat{U}(t,s) : G(E) \to G(E)$ is a uniformly bounded operator with

$$\left\|\widehat{U}(t,s)\right\| \leqslant M.$$

Next, we show that $\widehat{U}(t,s)$ is strongly and uniformly continuous: Let $\widehat{x} \in G(E)$ and $\varepsilon > 0$ be given. Choose $N \in \mathbb{N}$, such that for all $n \geqslant N$

$$\|x_n - x\| \leqslant \frac{\varepsilon}{M5} \quad \text{and} \quad \|[U_n(t,s) - U_N(t,s)]x\| \leqslant \frac{\varepsilon}{5} \quad \forall\ (t,s) \in \triangle.$$

Since the U_n are strongly and uniformly continuous, we can choose $\delta = \delta(\varepsilon) > 0$, such that

$$\|[U_N(t_1,s_1) - U_N(t_2,s_2)]x\| \leqslant \frac{\varepsilon}{5} \quad \text{and} \quad \max_{1\leqslant n\leqslant N-1} \|[U_n(t_1,s_1) - U_n(t_2,s_2)]x_n\| \leqslant \varepsilon,$$

for all $\|(t_1,s_1) - (t_2,s_2)\| \leqslant \delta$.

Case 1: $n \geqslant N$, $\|(t_1,s_1) - (t_2,s_2)\| \leqslant \delta$

$$\begin{aligned}\big\|[U_n(t_1,s_1) - U_n(t_2,s_2)]x_n\big\| &\leqslant \|U_n(t_1,s_1)\| \, \|x_n - x\| + \|[U_n(t_1,s_1) - U_N(t_1,s_1)]\, x\| \\ &+ \|[U_N(t_1,s_1) - U_N(t_2,s_2)]\, x\| + \|[U_N(t_2,s_2) - U_n(t_2,s_2)]\, x\| \\ &+ \|U_n(t_2,s_2)\| \, \|x - x_n\| \leqslant M\frac{\varepsilon}{M5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + M\frac{\varepsilon}{M5} = \varepsilon\end{aligned}$$

Case 2: $n < N$, $\|(t_1,s_1) - (t_2,s_2)\| \leqslant \delta$:

$$\|[U_n(t_1,s_1) - U_n(t_2,s_2)]\, x_n\| \leqslant \max_{1\leqslant n\leqslant N-1} \|[U_n(t_1,s_1) - U_n(t_2,s_2)]x_n\| \leqslant \varepsilon$$

Taking both cases together, gives

$$\left\|\left[\widehat{U}(t_1,s_1) - \widehat{U}(t_2,s_2)\right]\widehat{x}\right\| \leqslant \varepsilon \qquad \forall\, \|(t_1,s_1) - (t_2,s_2)\| \leqslant \delta, \tag{4.2}$$

i.e. $\widehat{U}$ is strongly and uniformly continuous. We define $\widehat{B}(t)\widehat{x} := (B(t)x_n)_{n\in\mathbb{N}}$. $\widehat{B}$ is uniformly bounded with

$$\left\|\widehat{B}(t)\right\| \leqslant C.$$

Next, we construct a solution $\widehat{H}(t,s)$ of the Volterra type integral equation in $G(E)$

$$\widehat{H}(t,s) = \widehat{U}(t,s) + \int_s^t \widehat{U}(t,\tau)\widehat{B}(\tau)\widehat{H}(\tau,s)d\tau$$

by the method of successive approximation.[5] We define

$$\widehat{R}_0(t,s) := \widehat{U}(t,s), \quad \widehat{R}_n(t,s) := \int_s^t \widehat{U}(t,\tau)\widehat{B}(\tau)\widehat{R}_{n-1}(\tau,s)d\tau, \quad \forall\, n \in \mathbb{N}.$$

i) $\left\|\widehat{R}_n(t,s)\right\| \leqslant M^{n+1}C^n\frac{(t-s)^n}{n!}$: We prove the assertion by induction over n.

- $n = 0$: $\left\|\widehat{R}_0(t,s)\right\| = \left\|\widehat{U}(t,s)\right\| \leqslant M$.
- $n \Rightarrow n+1$:

$$\begin{aligned}\left\|\widehat{R}_{n+1}(t,s)\right\| &\leqslant \int_s^t \left\|\widehat{U}(t,\tau)\right\| \left\|\widehat{B}(\tau)\right\| \left\|\widehat{R}_n(\tau,s)\right\| d\tau \\ &\leqslant MCM^{n+1}C^n \int_s^t \frac{(\tau-s)^n}{n!} d\tau = M^{n+2}C^{n+1}\frac{(t-s)^{n+1}}{(n+1)!}\end{aligned}$$

ii) $\widehat{R}_n(t,s)$ is strongly continuous in t:

- $n = 0$: $\widehat{R}_0(t,s) = \widehat{U}(t,s)$ which is strongly continuous (see (4.2)).
- $n \in \mathbb{N}$: Let $\widehat{x} \in G(E)$ be given. Let $(t,s) \in \triangle$, $\tau \in [s,t]$ and $t_k \in [t,T], k \in \mathbb{N}$, such that $t_k \to t$.

$$\begin{aligned} f_k(\tau) &:= \widehat{U}(t_k,\tau)\widehat{B}(\tau)\widehat{R}_{n-1}(\tau,s)\widehat{x} \\ \|f_k(\tau)\| &\leqslant \frac{M^2C(MCT)^{n-1}}{(n-1)!}\|\widehat{x}\| \\ f_k(\tau) &\overset{k\to\infty}{\longrightarrow} \widehat{U}(t,\tau)\widehat{B}(\tau)\widehat{R}_{n-1}(\tau,s)\widehat{x} \qquad \text{since } \widehat{U} \text{ is strongly continuous.}\end{aligned}$$

  Applying the dominated convergence theorem, gives

$$\begin{aligned}\lim_{k\to\infty}\int_s^t \widehat{U}(t_k,\tau)\widehat{B}(\tau)\widehat{R}_{n-1}(\tau,s)\widehat{x}d\tau &= \lim_{k\to\infty}\int_s^t f_k(\tau)d\tau = \int_s^t \lim_{k\to\infty} f_k(\tau)d\tau \\ &= \int_s^t \widehat{U}(t,\tau)\widehat{B}(\tau)\widehat{R}_{n-1}(\tau,s)\widehat{x}d\tau.\end{aligned}$$

[5]This is the part that has not been carried out in [Kre71]. Krein refers to Theorem 1.6.4 in [Kis64] page 295 where the strong continuity of $\widehat{H}_n(t,s)$ in t is proved.

This gives

$$\begin{aligned}
&\widehat{R}_n(t_k,s)\widehat{x} = \int_s^{t_k} \widehat{U}(t_k,\tau)\widehat{B}(\tau)\widehat{R}_{n-1}(\tau,s)\widehat{x}d\tau \\
&= \int_s^{t} \widehat{U}(t_k,\tau)\widehat{B}(\tau)\widehat{R}_{n-1}(\tau,s)\widehat{x}d\tau + \int_t^{t_k} \widehat{U}(t_k,\tau)\widehat{B}(\tau)\widehat{R}_{n-1}(\tau,s)\widehat{x}d\tau \\
&\overset{\theta_k \text{ between } t \text{ and } t_k}{=} \int_s^{t} \widehat{U}(t_k,\tau)\widehat{B}(\tau)\widehat{R}_{n-1}(\tau,s)\widehat{x}d\tau + (t_k - t)\widehat{U}(t_k,\theta_k)\widehat{B}(\theta_k)\widehat{R}_{n-1}(\theta_k,s)\widehat{x} \\
&\overset{k\to\infty}{\longrightarrow} \int_s^{t} \widehat{U}(t,\tau)\widehat{B}(\tau)\widehat{R}_{n-1}(\tau,s)\widehat{x}d\tau + 0 = \widehat{R}_n(t,s)\widehat{x}.
\end{aligned}$$

iii) $\widehat{R}_n(t,s)$ is strongly continuous in s:

If $\widehat{x} \in G(E)$, $(t,s) \in \triangle$, $k \in (0, t-s)$ and $\left\|\left[\widehat{U}(t,s+k) - \widehat{U}(t,s)\right]\widehat{x}\right\| \leqslant \varepsilon$, then

$$\left\|\left[\widehat{R}_n(t,s+k) - \widehat{R}_n(t,s)\right]\widehat{x}\right\| \leqslant \frac{(MC)^n}{n!}\left[\varepsilon(t-s)^n + M\left\|\widehat{x}\right\|\left[(t-s+k)^n - (t-s)^n\right]\right] :$$

Proof by induction over n:

- $n = 0$: $\left\|\left[\widehat{R}_0(t,s+k) - \widehat{R}_0(t,s)\right]\widehat{x}\right\| = \left\|\left[\widehat{U}(t,s+k) - \widehat{U}(t,s)\right]\widehat{x}\right\| \leqslant \varepsilon$
- $n \Rightarrow n+1$:

$$\begin{aligned}
&\left\|\left[\widehat{R}_{n+1}(t,s+k) - \widehat{R}_{n+1}(t,s)\right]\widehat{x}\right\| \\
&= \left\|\int_{s+k}^{t} \widehat{U}(t,\tau)\widehat{B}(\tau)\widehat{R}_n(\tau,s+k)\widehat{x}\, d\tau - \int_s^t \widehat{U}(t,\tau)\widehat{B}(\tau)\widehat{R}_n(\tau,s)\widehat{x}\, d\tau\right\| \\
&\leqslant \int_s^t \left\|\widehat{U}(t,\tau)\right\|\left\|\widehat{B}(\tau)\right\|\left\|\left[\widehat{R}_n(\tau,s+k) - \widehat{R}_n(\tau,s)\right]\widehat{x}\right\| d\tau \\
&\quad + \int_s^{s+k} \left\|\widehat{U}(t,\tau)\right\|\left\|\widehat{B}(\tau)\right\|\left\|\widehat{R}_n(\tau,s)\right\|\left\|\widehat{x}\right\| d\tau \\
&\leqslant \frac{(MC)^{n+1}}{n!}\int_s^t \left[\varepsilon(\tau-s)^n + M\left\|\widehat{x}\right\|\left[(\tau-s+k)^n - (\tau-s)^n\right]\right] d\tau \\
&\quad + (MC)^{n+1}M\left\|\widehat{x}\right\|\int_s^{s+k} \frac{(\tau-s)^n}{n!} d\tau \\
&= \frac{(MC)^{n+1}}{n!}\left[\varepsilon\frac{(t-s)^{n+1}}{n+1} + M\left\|\widehat{x}\right\|\left[\frac{(t-s+k)^{n+1} - k^{n+1}}{n+1} - \frac{(t-s)^{n+1}}{n+1}\right]\right. \\
&\quad \left. + M\left\|\widehat{x}\right\|\frac{k^{n+1}}{n+1}\right] \\
&= \frac{(MC)^{n+1}}{(n+1)!}\left[\varepsilon(t-s)^{n+1} + M\left\|\widehat{x}\right\|\left[(t-s+k)^{n+1} - (t-s)^{n+1}\right]\right]
\end{aligned}$$

Next, we add up the $\widehat{R}_n$ to get the approximation operators

$$\widehat{H}_n(t,s) := \sum_{k=0}^{n} \widehat{R}_k(t,s).$$

The properties of the $\widehat{R}_n$ imply the following:

- $\left\|\widehat{H}_n(t,s)\right\| \leqslant Me^{MCT}$
- $\widehat{H}_n(t,s)$ is strongly continuous in $\triangle$ since it is the finite sum of strongly continuous operators.
- $\widehat{H}_n(t,s)$ converges uniformly with respect to t and s to a bounded operator $\widehat{H}(t,s)$: Let $n > m \geqslant N$

  $$\left\|\widehat{H}_n(t,s) - \widehat{H}_m(t,s)\right\| \leqslant \sum_{k=m+1}^{n} \left\|\widehat{R}_k(t,s)\right\| \leqslant M \sum_{k=m+1}^{n} \frac{(MCT)^k}{k!} \leqslant M\frac{(MCT)^{N+1}}{(N+1)!}e^{MCT}.$$

  Since $\mathcal{L}(G(E))$ is a Banach space, $\widehat{H}_n(t,s)$ converges to a bounded operator

  $$\widehat{H}(t,s) \in \mathcal{L}(G(E)).$$

  The convergence is uniform as can be seen from the Cauchy sequence estimates and thus, $\widehat{H}(t,s)$ is strongly continuous in $\triangle$.

The operators $\widehat{U}$ and $\widehat{B}$ operate on each element of $\widehat{x}$ independently. By the construction process above it can be seen that $\widehat{H}$ has the same property, i.e.

$$\widehat{H}(t,s)\widehat{x} =: (H_n(t,s)x_n)_{n\in\mathbb{N}}.$$

Furthermore, $H_n(t,s)$ satisfies Equation (4.1). Since the solution of (4.1) is unique, this implies $H_n(t,s) = \widetilde{U}_n(t,s)$. For $x \in E$ we define $\widehat{x} := (x)_{n\in\mathbb{N}} \in G(E)$. From the continuity of $\widehat{H}(t,s)\widehat{x}$ in $G(E)$ follows that the $\widetilde{U}_n(t,s)x$ are equicontinuous for $(t,s) \in \triangle$. $\widehat{H}(t,s)\widehat{x} \in G(E)$ implies that $\widetilde{U}_n(t,s)x$ converges for fixed $(t,s) \in \triangle$. The strong convergence, the equicontinuity and the compactness of $\triangle$ imply that $\widetilde{U}_n(t,s)$ converges strongly and uniformly with respect to $(t,s) \in \Delta$. □

Lemma 4.2.5. *Let $A(t)$ be a strongly continuously differentiable family of operators with common dense domain $\mathcal{D}(A)$. Assume further that $A(t)$ has a bounded inverse and satisfies*

$$\left\|(A(t) - \lambda\mathbb{1})^{-1}\right\|_{\mathcal{L}(E)} \leqslant \frac{1}{\lambda}, \quad \forall\ \lambda > 0.$$

The operators

$$A_n(t) := -nA(t)\left(A(t) - n\right)^{-1} = -n - n^2\left(A(t) - n\right)^{-1}, \qquad \text{for all } n \in \mathbb{N},$$

have the following properties:[6]

[6]These operators have been introduced on page 204 in [Kre71].

1. *$A_n(t)$ is bounded and strongly continuous.*
2. *$A_n(t)A^{-1}(t)$ converges strongly and uniformly in t to $\mathbb{1}$.*[7]
3. *$\left\|(A_n(t)-\lambda)^{-1}\right\| \leqslant \frac{1}{\lambda}$ for $\lambda > 0$.*[8]
4. *The evolution operators $U_n(t,s)$ of $A_n(t)$ satisfy $\|U_n(t,s)\| \leqslant 1$.*[9]

Proof. Ad 1: $A_n(t)$ is closed and $\mathcal{D}(A_n(t)) = E$. Thus, the operator is bounded by the closed graph theorem. Since $A(t)$ is strongly continuous in t, we get for $x \in E$

$$\begin{aligned}\|[A_n(s)-A_n(t)]\,x\| &= \left\|\left[-n-n^2\,(A(s)-n)^{-1}+n+n^2\,(A(t)-n)^{-1}\right]x\right\| \\ &= n^2\left\|(A(s)-n)^{-1}\,(A(s)-A(t))\,(A(t)-n)^{-1}\,x\right\| \\ &\leqslant n\left\|(A(s)-A(t))\,(A(t)-n)^{-1}\,x\right\| \overset{|s-t|\to 0}{\longrightarrow} 0,\end{aligned}$$

i.e. $A_n(t)$ is strongly continuous in t.

Ad 2: $A_n(t)A^{-1}(t)$ converges strongly and uniformly in t to $\mathbb{1}$:

- For $x_0 \in \mathcal{D}(A)$ we find

$$\begin{aligned}\|\,[A_n(t)-A(t)]\,A^{-1}(t)x_0\| &= \left\|\left[-n\,(A(t)-n)^{-1}-\mathbb{1}\right]x_0\right\| \\ &= \left\|\left[(A(t)-n)^{-1}\,(A(t)-n-A(t))-\mathbb{1}\right]x_0\right\| = \left\|(A(t)-n)^{-1}\right\|\|A(t)x_0\| \\ &\overset{A\text{ strongly continuous}}{\leqslant} \frac{1}{n}\max_{t\in[0,T]}\|A(t)x_0\|\,.\end{aligned}$$

- Let $x \in E$. Since $\mathcal{D}(A)$ is dense in E, there is a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathcal{D}(A)$ that converges to x. Let $\varepsilon > 0$ and choose $m \in \mathbb{N}$, such that $\|x - x_m\| \leqslant \frac{\varepsilon}{4}$. Then choose $n \in \mathbb{N}$, such that $\frac{1}{n}\max_{t\in[0,T]}\|A(t)x_m\| \leqslant \frac{\varepsilon}{2}$:

$$\begin{aligned}&\left\|\,[A_n(t)-A(t)]A^{-1}(t)x\right\| \\ &\qquad\leqslant (\underbrace{\left\|A_n(t)A^{-1}(t)\right\|}_{=\|n(A(t)-n)^{-1}\|\leqslant 1}+\|\mathbb{1}\|)\,\|x-x_m\| + \left\|[A_n(t)-A(t)]\,A^{-1}(t)x_m\right\| \\ &\qquad\leqslant 2\,\|x-x_m\| + \frac{1}{n}\max_{t\in[0,T]}\|A(t)x_m\| \leqslant \varepsilon\end{aligned}$$

[7]This has been proved on the top of page 205 in [Kre71].
[8]This has been shown in the proof of Lemma 3.2 on page 205 in [Kre71].
[9]This has been shown in the proof of Theorem 3.10 on page 203 f. in [Kre71].

Ad 3: $\left\|(A_n(t)-\lambda)^{-1}\right\| \leqslant \frac{1}{\lambda}$ for $\lambda > 0$:

$$\begin{aligned}
(A_n(t)-\lambda)^{-1} &= \Big(-nA(t)\,(A(t)-n)^{-1}-\lambda\Big)^{-1}\\
&= (A(t)-n)\,\Big(-nA(t)-(A(t)-n)\,\lambda\Big)^{-1}\\
&= -\frac{1}{n+\lambda}\,(A(t)-n)\left(A(t)-\frac{n\lambda}{n+\lambda}\right)^{-1}\\
&= -\frac{1}{n+\lambda}\left(A(t)-\frac{n\lambda}{n+\lambda}+n\left(\frac{\lambda}{n+\lambda}-1\right)\right)\left(A(t)-\frac{n\lambda}{n+\lambda}\right)^{-1}\\
&= -\frac{1}{n+\lambda}\mathbb{1}+\frac{n^2}{(n+\lambda)^2}\left(A(t)-\frac{n\lambda}{n+\lambda}\right)^{-1}
\end{aligned}$$

This implies

$$\left\|(A_n(t)-\lambda)^{-1}\right\| \leqslant \frac{1}{n+\lambda}+\frac{n^2}{(n+\lambda)^2}\frac{n+\lambda}{n\lambda}=\frac{1}{\lambda}.$$

Ad 4: $\|U_n(t,s)\| \leqslant 1$: Due to Lemma 4.2.3, the operators A_n have evolution operators $U_n(t,s)$.

- In 1. we proved that $A_n(t)$ is strongly continuous and bounded on the compact interval $[0,T]$. For $x \in E$ this implies $\sup_{t\in[0,T]}\|A_n(t)\| < \infty$. Let $\varepsilon_k > 0$ with $\varepsilon_k \overset{k\to\infty}{\longrightarrow} 0$ and $x \in E$
$$\left\|A_n(t)\left(A_n(t)-\frac{1}{\varepsilon_k}\right)^{-1}\right\| \overset{3.}{\leqslant} \varepsilon_k \sup_{t\in[0,T]}\|A_n(t)\| \overset{k\to\infty}{\longrightarrow} 0,$$
i.e. $A_n(t)\left(A_n(t)-\frac{1}{\varepsilon_k}\right)^{-1}$ converges strongly and uniformly in t to 0.

- The derivative of the function $\phi(t) := \|U_n(t,s)x\|$ is non-positive:
$U_n(t,s)$ is strongly differentiable in t, i.e.
$$\frac{U_n(t+\varepsilon_k,s)x-U_n(t,s)x}{\varepsilon_k} = A_n(t)U_n(t,s)x+o(\varepsilon_k).$$
This implies:
$$\begin{aligned}
\frac{U_n(t+\varepsilon_k,s)}{\varepsilon_k}x &= \left(A_n(t)+\frac{1}{\varepsilon_k}\right)U_n(t,s)x+o(\varepsilon_k)\\
&= \left(A_n^2(t)-\frac{1}{\varepsilon_k^2}\right)\left(A_n(t)-\frac{1}{\varepsilon_k}\right)^{-1}U_n(t,s)x+o(\varepsilon_k)\\
&= -\frac{1}{\varepsilon_k^2}\left(A_n(t)-\frac{1}{\varepsilon_k}\right)^{-1}U_n(t,s)x\\
&\quad + A_n(t)\left(A_n(t)-\frac{1}{\varepsilon_k}\right)^{-1}A_n(t)U_n(t,s)x+o(\varepsilon_k)
\end{aligned}$$

$$\overset{3.}{\Rightarrow} \quad \frac{\|U_n(t+\varepsilon_k,s)x\|}{\varepsilon_k} \leqslant \frac{\|U_n(t,s)x\|}{\varepsilon_k} + \left\| A_n(t)\left(A_n(t) - \frac{1}{\varepsilon_k}\right)^{-1} A_n(t)U_n(t,s)x \right\| + o(\varepsilon_k)$$

$$\Rightarrow \quad \frac{\|U_n(t+\varepsilon_k,s)x\| - \|U_n(t,s)x\|}{\varepsilon_k} \xrightarrow{k\to\infty} 0$$

$$\Rightarrow \quad \phi'(t) = \lim_{k\to\infty} \frac{\|U_n(t+\varepsilon_k,s)x\| - \|U_n(t,s)x\|}{\varepsilon_k} \leqslant 0$$

$$\Rightarrow \quad \|U_n(t,s)x\| \leqslant \|U_n(s,s)x\| = \|x\| \quad \Rightarrow \quad \|U_n(t,s)\| \leqslant 1$$

□

Theorem 4.2.6. [10] *If $\mathcal{D}(A) \subset E$ is dense and if $A(t) : \mathcal{D}(A) \to E$ is strongly continuously differentiable in $[0,T]$, has a bounded inverse and satisfies*

$$\left\|(A(t) - \lambda \mathbb{1})^{-1}\right\|_{\mathcal{L}(E)} \leqslant \frac{1}{\lambda}, \quad \forall\, \lambda > 0 \text{ and } t \in [0,T], \tag{4.3}$$

there exists a family of operators $U(t,s) \in \mathcal{L}(E)$ for $(t,s) \in \triangle$ with the properties:

1. *$\|U(t,s)\|_{\mathcal{L}(E)} \leqslant 1$ for all $(t,s) \in \triangle$.*
2. *$U(t,s)$ is strongly continuous for all $(t,s) \in \triangle$.*
3. *$U(t,s)(\mathcal{D}(A)) \subset \mathcal{D}(A)$ and the operators $A(t)U(t,s)A^{-1}(s)$ are bounded and strongly continuous for all $(t,s) \in \triangle$.*
4. *$U(t,s)$ is strongly differentiable on $\mathcal{D}(A)$ relative to t and s and*

$$\frac{\partial}{\partial t}U(t,s) = A(t)U(t,s) \quad \text{and} \quad \frac{\partial}{\partial s}U(t,s) = -U(t,s)A(s). \tag{4.4}$$

5. *$U(t,\tau)U(\tau,s) = U(t,s)$, $U(t,t) = \mathbb{1}$, for all $s,\tau,t \in \mathbb{R}$ with $0 \leqslant s \leqslant \tau \leqslant t \leqslant T$.*

Proof. The key step in the proof is to show that the evolution operators $U_n(t,s)$ to the approximation operators $A_n(t)$, as defined in Lemma 4.2.5, converge strongly and uniformly in (t,s) to an operator $U(t,s)$.[11]

For $N \in \mathbb{N}$ we divide $[0,T]$ into pieces of length TN^{-1} and define the piecewise constant operators:

$$\widetilde{A}_{n,N}(t) = A_n\left(\frac{k-1}{N}T\right), \text{ for } t \in \left[\frac{k-1}{N}T, \frac{k}{N}T\right), \ (k = 1,2,\dots,N).$$

[10] Essentially, this is Theorem 3.11 on page 208 in [Kre71]. There it is assumed that

$$\left\|(A(t) - \lambda \mathbb{1})^{-1}\right\|_{\mathcal{L}(E)} \leqslant \frac{1}{1+\lambda}, \quad \forall\, \lambda \geqslant 0.$$

This estimate implies for $\lambda = 0$ that $A(t)$ has a bounded inverse and for $\lambda > 0$ the inequality assumed here. To demonstrate that the more restrictive assumption is obsolete, the full proof is presented here.

[11] This step is equal to the proof of Theorem 3.11 on page 208 in [Kre71].

Their evolution operators are:

$$\widetilde{U}_{n,N}(t,s) = \exp\left[\left(t - \frac{j-1}{N}T\right) A_n\left(\frac{j-1}{N}T\right)\right] \exp\left[\frac{T}{N} A_n\left(\frac{j-2}{N}T\right)\right] \\ \cdots \exp\left[\frac{T}{N} A_n\left(\frac{i}{N}T\right)\right] \exp\left[\left(\frac{i}{N}T - s\right) A_n\left(\frac{i-1}{N}T\right)\right]$$

if there are natural numbers $i < j$ satisfying

$$\frac{i-1}{N}T \leqslant s < \frac{i}{N}T \leqslant \frac{j-1}{N}T \leqslant t < \frac{j}{N}T$$

and if there is an $i \in \mathbb{N}$, such that $\frac{i-1}{N}T \leqslant s < t < \frac{i}{N}T$, then

$$\widetilde{U}_{n,N}(t,s) = \exp\left[(t-s)\, A_n\left(\frac{i-1}{N}T\right)\right].$$

We estimate the norm of the product terms appearing in $\widetilde{U}_{n,N}$:

$$\left\|e^{tA_n(\tau)}\right\| = \left\|e^{-nt-n^2t(A(\tau)-n)^{-1}}\right\| = \left\|e^{-nt}\sum_{k=0}^{\infty} \frac{(-n^2t)^k\,(A(\tau)-n)^{-k}}{k!}\right\| \\ \leqslant e^{-nt}\sum_{k=0}^{\infty}\frac{(n^2t)^k}{k!}\underbrace{\left\|(A(\tau)-n)^{-1}\right\|^k}_{\overset{(4.3)}{\leqslant} n^{-k}} = e^{-nt+nt} = 1.$$

This implies

$$\left\|\widetilde{U}_{n,N}(t,s)\right\| \leqslant 1 \text{ for all } (t,s) \in \triangle.$$

Let $x_0 \in \mathcal{D}(A)$ and $\varepsilon > 0$. The difference between two product terms is:

$$\left\|\left[e^{(t-s)A_n(\tau)} - e^{(t-s)A_m(\tau)}\right] x_0\right\| = \left\|-\int_s^t \frac{\partial}{\partial u}\left[e^{(t-u)A_n(\tau)}e^{(u-s)A_m(\tau)}\right] du x_0\right\| \\ = \left\|\int_s^t e^{(t-u)A_n(\tau)}(A_n(\tau) - A_m(\tau))e^{(u-s)A_m(\tau)} du x_0\right\| \\ \leqslant (t-s)\left\|\left[A_n(\tau) - A_m(\tau)\right] x_0\right\| \\ \leqslant T \sup_{u\in[0,T]}\left\|\left[A_n(u) - A_m(u)\right] A^{-1}(u)\right\| \sup_{v\in[0,T]}\|A(v)x_0\| \leqslant \frac{\varepsilon}{2T}, \quad \forall\, n,m \geqslant n_0 = n_0(\varepsilon, x_0)$$

since $A_n(t)A^{-1}(t)$ converges strongly and uniformly to $\mathbb{1}$ (Lemma 4.2.5, 2.) and $A(t)x_0$ is bounded uniformly in t. This implies

$$\|\widetilde{U}_{n,N}(t,s)x_0 - \widetilde{U}_{m,N}(t,s)x_0\| \\ \leqslant \left\|\left(\exp\left[\left(t - \frac{j-1}{N}T\right) A_n\left(\frac{j-1}{N}T\right)\right] - \exp\left[\left(t - \frac{j-1}{N}T\right) A_m\left(\frac{j-1}{N}T\right)\right]\right) x_0\right\| \\ + \sum_{l=i}^{j-2}\left\|\left(\exp\left[\frac{T}{N}A_n\left(\frac{l}{N}T\right)\right] - \exp\left[\frac{T}{N}A_m\left(\frac{l}{N}T\right)\right]\right) x_0\right\| \\ + \left\|\left(\exp\left[\left(\frac{i}{N}T - s\right) A_n\left(\frac{i-1}{N}T\right)\right] - \exp\left[\left(\frac{i}{N}T - s\right) A_m\left(\frac{i-1}{N}T\right)\right]\right) x_0\right\|$$

$$
\begin{aligned}
&\leqslant \left(t-\frac{j-1}{N}T\right)\left\|\left[A_n\left(\frac{j-1}{N}T\right)-A_m\left(\frac{j-1}{N}T\right)\right]x_0\right\| \\
&\quad+\sum_{l=i}^{j-2}\frac{T}{N}\left\|\left[A_n\left(\frac{l}{N}T\right)-A_m\left(\frac{l}{N}T\right)\right]x_0\right\| \\
&\quad+\left(\frac{i}{N}T-s\right)\left\|\left[A_n\left(\frac{i-1}{N}T\right)-A_m\left(\frac{i-1}{N}T\right)\right]x_0\right\| \\
&\leqslant N\frac{T}{N}\frac{\varepsilon}{2T}=\frac{\varepsilon}{2}, \qquad \forall\, n,m\geqslant n_0=n_0(\varepsilon,x_0).
\end{aligned}
$$

For $x\in E$ choose $x_0\in\mathcal{D}(A)$ with $\|x-x_0\|\leqslant\frac{\varepsilon}{4}$.

$$
\begin{aligned}
\left\|\left[\widetilde{U}_{n,N}(t,s)-\widetilde{U}_{m,N}(t,s)\right]x\right\| &\leqslant \left(\left\|\widetilde{U}_{n,N}(t,s)\right\|+\left\|\widetilde{U}_{m,N}(t,s)\right\|\right)\|x-x_0\| \\
&\quad+\left\|\left[\widetilde{U}_{n,N}(t,s)-\widetilde{U}_{m,N}(t,s)\right]x_0\right\|\leqslant\varepsilon \quad \forall\, n,m\geqslant n_0=n_0(\varepsilon,x)
\end{aligned}
$$

In summary, we have proved for $x\in E$ and $\varepsilon>0$:

$$
\boxed{\left\|\left[\widetilde{U}_{n,N}(t,s)-\widetilde{U}_{m,N}(t,s)\right]x\right\|\leqslant\varepsilon \qquad \forall\, n,m\geqslant n_0(\varepsilon,x),\ (t,s)\in\triangle,\ \text{ and } N\in\mathbb{N}.} \tag{4.5}
$$

Next, we calculate the difference between the approximation operators U_n and $\widetilde{U}_{n,N}$.

$$
\begin{aligned}
\widetilde{U}_{n,N}(t,s)-U_n(t,s) &= \widetilde{U}_{n,N}(t,s)U_n(s,s)-\widetilde{U}_{n,N}(t,t)U_n(t,s) \\
&= -\int_s^t\frac{\partial}{\partial\tau}\left[\widetilde{U}_{n,N}(t,\tau)U_n(\tau,s)\right]d\tau \\
&= \int_s^t\left[\widetilde{U}_{n,N}(t,\tau)\widetilde{A}_{n,N}(\tau)U_n(\tau,s)-\widetilde{U}_{n,N}(t,\tau)A_n(\tau)U_n(\tau,s)\right]d\tau \\
&= \int_s^t\widetilde{U}_{n,N}(t,\tau)\left[\widetilde{A}_{n,N}(\tau)-A_n(\tau)\right]U_n(\tau,s)d\tau
\end{aligned}
$$

For $x_0\in\mathcal{D}(A)$ the formula above implies:

$$
\begin{aligned}
&\left\|\left[\widetilde{U}_{n,N}(t,s)-U_n(t,s)\right]x_0\right\| \qquad (4.6)\\
&\quad\leqslant T\max_{\tau\in[0,T]}\left\|\left[\widetilde{A}_{n,N}(\tau)-A_n(\tau)\right]A^{-1}(\tau)\right\|\max_{\tau\in[0,T]}\left\|A(\tau)U_n(\tau,s)A^{-1}(s)\right\|\|A(s)x_0\|.
\end{aligned}
$$

We proceed by proving estimates for the three terms, separately:

- First term: Given $\tau \in [0,T]$, choose $k \in \mathbb{N}$, such that $\frac{k-1}{N}T \leqslant \tau < \frac{k}{N}T$:

$$\begin{aligned}
&\left\| \left[\widetilde{A}_{n,N}(\tau) - A_n(\tau)\right] A^{-1}(\tau)\right\| = \left\| \left[A_n\left(\frac{k-1}{N}T\right) - A_n(\tau)\right] A^{-1}(\tau)\right\| \\
&\overset{\text{def. of } A_n}{=} \left\| \left[-n - n^2\left(A\left(\frac{k-1}{N}T\right) - n\right)^{-1} + n + n^2\left(A(\tau)-n\right)^{-1}\right] A^{-1}(\tau)\right\| \\
&\leqslant n \underbrace{\left\|\left(A\left(\frac{k-1}{N}T\right) - n\right)^{-1}\right\|}_{\overset{(4.3)}{\leqslant} \frac{1}{n}} \left\| \left[A\left(\frac{k-1}{N}T\right) - A(\tau)\right] A^{-1}(\tau)\right\| n \underbrace{\left\|\left(A(\tau)-n\right)^{-1}\right\|}_{\overset{(4.3)}{\leqslant} \frac{1}{n}} \\
&\leqslant \left\| \left[A\left(\frac{k-1}{N}T\right) - A(\tau)\right] A^{-1}(\tau)\right\| = \left\| \int_{\frac{k-1}{N}T}^{\tau} A'(t)A^{-1}(\tau)dt\right\| \\
&\leqslant \sup_{(t,\tau)\in[0,T]^2} \left\|A'(t)A^{-1}(\tau)\right\| \frac{T}{N} =: \frac{C_2 T}{N}.
\end{aligned}$$

- Second term: Consider the sequence of differential equations

$$\frac{\partial}{\partial t} = A_n(t) + A'(t)A^{-1}(t), \qquad n \in \mathbb{N}. \tag{4.7}$$

The operators on the right are strongly continuous and bounded. Due to Lemma 4.2.3, this differential equation has evolution operators $V_n(t,s)$. The computations

$$\begin{aligned}
\frac{\partial}{\partial t}\left(A(t)U_n(t,s)A^{-1}(s)\right) &= A'(t)U_n(t,s)A^{-1}(s) + A(t)A_n(t)U_n(t,s)A^{-1}(s) \\
&\overset{[A(t),A_n(t)]=0}{=} \left(A_n(t) + A'(t)A^{-1}(t)\right)\left(A(t)U_n(t,s)A^{-1}(s)\right) \\
A(t)U_n(t,t)A^{-1}(t) &= A(t)A^{-1}(t) = \mathbb{1}
\end{aligned}$$

and uniqueness imply $V_n(t,s) = A(t)U_n(t,s)A^{-1}(s)$. The differential equations (4.7) satisfy the assumptions of Theorem 4.2.4. Thus,

$$\|U_n(t,s)\| \leqslant 1$$

implies the uniform boundedness of $V_n(t,s)$ in t, s and n. Therefore,

$$\left\|A(t)U_n(t,s)A^{-1}(s)\right\| = \|V_n(t,s)\| \leqslant \sup_{0\leqslant s\leqslant t\leqslant T,\ n\in\mathbb{N}} \|V_n(t,s)\| =: C_3 < \infty.$$

- Third term:

$$\|A(s)x_0\| \leqslant \max_{s\in[0,T]} \|A(s)x_0\| =: C_4(x_0)$$

Inserting the three estimates in (4.6), gives

$$\left\| \left[\widetilde{U}_{n,N}(t,s) - U_n(t,s)\right] x_0\right\| \leqslant C_2 C_3 C_4(x_0) \frac{T^2}{N}.$$

Since $\left\|\widetilde{U}_{n,N}(t,s)\right\| \leqslant 1$ and $\mathcal{D}(A)$ is dense in E, this implies for $x \in E$ and $\varepsilon > 0$:

$$\boxed{\left\|\left[\widetilde{U}_{n,N}(t,s) - U_n(t,s)\right]x\right\| \leqslant \varepsilon, \qquad \forall\ N \geqslant N_0 = N_0(\varepsilon, x).} \tag{4.8}$$

Let $x \in E$ and $\varepsilon > 0$ be given. Due to the framed Formula (4.8), we can choose $N_0 = N_0(\varepsilon, x) \in \mathbb{N}$, such that

$$\left\|\left[\widetilde{U}_{n,N}(t,s) - U_n(t,s)\right]x\right\| \leqslant \frac{\varepsilon}{3}, \qquad \forall\ N \geqslant N_0(\varepsilon, x)$$

and due to the framed Formula (4.5) it exists an $n_0 = n_0(\varepsilon, x) \in \mathbb{N}$, such that

$$\left\|\left[\widetilde{U}_{n,N}(t,s) - \widetilde{U}_{m,N}(t,s)\right]x\right\| \leqslant \frac{\varepsilon}{3} \qquad \forall\ n, m \geqslant n_0(\varepsilon, x),\ \forall\ N \in \mathbb{N}.$$

The calculation

$$\begin{aligned}\|[U_n(t,s) - U_m(t,s)]x\| &\leqslant \left\|\left[U_n(t,s) - \widetilde{U}_{n,N}(t,s)\right]x\right\| + \left\|\left[\widetilde{U}_{n,N}(t,s) - \widetilde{U}_{m,N}(t,s)\right]x\right\| \\ &\quad + \left\|\left[\widetilde{U}_{m,N}(t,s) - U_m(t,s)\right]x\right\| \leqslant \varepsilon \qquad \forall\ n, m \geqslant n_0(\varepsilon, x)\end{aligned}$$

shows that $U_n(t,s)x$ is a Cauchy sequence in the Banach space E. Thus, $U_n(t,s)$ converges strongly and uniformly to $U(t,s)$.

Next, we prove the asserted properties of $U(t,s)$.

1. $U(t,s)$ is linear and $\|U(t,s)\|_{\mathcal{L}(E)} \leqslant 1$ for $(t,s) \in \triangle$:
   Linearity follows from the linearity of U_n. Let $x \in E$

   $$\|U(t,s)x\| \leqslant \underbrace{\|[U(t,s) - U_n(t,s)]\,x\|}_{\xrightarrow{n\to\infty} 0} + \underbrace{\|U_n(t,s)x\|}_{\leqslant \|x\|} \quad \Rightarrow \quad \|U(t,s)\| \leqslant 1.$$

2. $U(t,s)$ is strongly continuous on E for $(t,s) \in \triangle$:

   Let $x \in E$, $(t_1, s_1) \in \triangle$ and $\varepsilon > 0$. Since U_n converges strongly and uniformly, we can choose an $n \in \mathbb{N}$, such that

   $$\|[U(t,s) - U_n(t,s)]\,x\| \leqslant \frac{\varepsilon}{3} \quad \forall\ (t,s) \in \triangle.$$

   Next, choose δ small enough, such that

   $$\|[U_n(t_1,s_1) - U_n(t_2,s_2)]\,x\| \leqslant \frac{\varepsilon}{3} \quad \forall\ (t_2,s_2) \in \triangle \text{ with } \|(t_1,s_1) - (t_2,s_2)\| \leqslant \delta.$$

   This implies

   $$\begin{aligned}\|[U(t_1,s_1) - U(t_2,s_2)]\,x\| &\leqslant \|[U(t_1,s_1) - U_n(t_1,s_1)]\,x\| + \|[U_n(t_1,s_1) - U_n(t_2,s_2)]\,x\| \\ &\quad + \|[U_n(t_2,s_2) - U(t_2,s_2)]\,x\| \leqslant \varepsilon,\end{aligned}$$

   for all $(t_2,s_2) \in \triangle$ with $\|(t_1,s_1) - (t_2,s_2)\| \leqslant \delta$.

3. $U(t,s)|_{\mathcal{D}(A)} : \mathcal{D}(A) \to \mathcal{D}(A)$ and the operator $A(t)U(t,s)A^{-1}(s)$ is bounded and strongly continuous in $\triangle$:

   We discussed above that V_n solves equation (4.7). The $U_n(t,s)$ are uniformly bounded relative to t,s,n and converge strongly and uniformly relative to t,s to a bounded operator U. By Theorem 4.2.4 this implies that the $V_n(t,s)$ are uniformly bounded relative to t,s,n and converge strongly and uniformly relative to t,s to an operator V. As for U this implies V is bounded and strongly continuous.

   For $x_0 \in \mathcal{D}(A)$ exists an $x \in E$, such that $x_0 = A^{-1}(s)x$,

$$U_n(t,s)A^{-1}(s)x \overset{n\to\infty}{\longrightarrow} U(t,s)A^{-1}(s)x$$

   and

$$A(t)U_n(t,s)A^{-1}(s)x = V_n(t,s)x \overset{n\to\infty}{\longrightarrow} V(t,s)x.$$

   Since $A(t)$ is closed, this implies that

$$U(t,s)x_0 = U(t,s)A^{-1}(s)x \in \mathcal{D}(A) \quad \text{and} \quad A(t)U(t,s)A^{-1}(s)x = V(t,s)x.$$

   Thus,

$$U(t,s)[\mathcal{D}(A)] \subset \mathcal{D}(A) \text{ and } V(t,s) = A(t)U(t,s)A^{-1}(s).$$

4. $U(t,s)$ is strongly differentiable on $\mathcal{D}(A)$ relative to t and s:
   Let $x_0 \in \mathcal{D}(A)$. Firstly,

$$\begin{aligned}
&\|[A_n(t)U_n(t,s) - A(t)U(t,s)]x_0\| \\
&\quad = \left\| \left[A_n(t)A^{-1}(t)A(t)U_n(t,s)A(s)^{-1}A(s) - A(t)U(t,s)A^{-1}(s)A(s)\right] x_0 \right\| \\
&\quad = \left\| \left[A_n(t)A^{-1}(t)V_n(t,s) - V(t,s)\right] A(s)x_0 \right\| \\
&\quad \leqslant \underbrace{\left\|A_n(t)A^{-1}(t)\right\|}_{\leqslant 1} \underbrace{\left\|\left[V_n(t,s) - V(t,s)\right] A(s)x_0\right\|}_{\overset{n\to\infty}{\longrightarrow} 0} \\
&\qquad\qquad + \underbrace{\left\| \left[A_n(t)A^{-1}(t) - \mathbb{1}\right] V(t,s)A(s)x_0 \right\|}_{\overset{n\to\infty}{\longrightarrow} 0 \ (4.2.5,\ 2.)} \overset{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

   This yields

$$\begin{aligned}
\frac{\partial}{\partial t}U(t,s)x_0 &= \frac{\partial}{\partial t}\lim_{n\to\infty} U_n(t,s)x_0 = \lim_{n\to\infty} \frac{\partial}{\partial t}U_n(t,s)x_0 \\
&= \lim_{n\to\infty} A_n(t)U_n(t,s)x_0 = A(t)U(t,s)x_0.
\end{aligned}$$

   Secondly,

$$\begin{aligned}
&\|[-U_n(t,s)A_n(s) + U(t,s)A(s)]x_0\| \\
&\quad \leqslant \underbrace{\|U_n(t,s)\|}_{\leqslant 1} \underbrace{\left\| \left[A_n(s)A^{-1}(s) - \mathbb{1}\right] A(s)x_0 \right\|}_{\overset{n\to\infty}{\longrightarrow} 0 \ (4.2.5,\ 2.)} + \underbrace{\left\|\left[U_n(t,s) - U(t,s)\right] A(s)x_0\right\|}_{\overset{n\to\infty}{\longrightarrow} 0} \overset{n\to\infty}{\longrightarrow} 0
\end{aligned}$$

gives

$$\begin{aligned}\frac{\partial}{\partial s}U(t,s)x_0 &= \frac{\partial}{\partial s}\lim_{n\to\infty} U_n(t,s)x_0 = \lim_{n\to\infty}\frac{\partial}{\partial s}U_n(t,s)x_0 \\ &= \lim_{n\to\infty}(-U_n(t,s)A(s)x_0) = -U(t,s)A(s)x_0.\end{aligned}$$

5. $U(t,\tau)U(\tau,s) = U(t,s),\ U(t,t) = \mathbb{1}$ for all $s,\tau,t \in \mathbb{R}$ with $0 \leqslant s \leqslant \tau \leqslant t \leqslant T$: $U_n(t,t) = \mathbb{1}$ and $U_n(t,s)x$ converges to $U(t,s)x$ for all $x \in E$. Thus,

$$U(t,t)x = \lim_{n\to\infty} U_n(t,t)x = x \ \forall x \in E \quad \Rightarrow \quad U(t,t) = \mathbb{1}, \quad \forall\, t \in [0,T].$$

Let $x_0 \in \mathcal{D}(A)$. It follows from 3. that $U(\tau,s)x_0 \in \mathcal{D}(A)$. Applying 4., gives

$$\begin{aligned}&\frac{\partial}{\partial \tau}\left[U(t,\tau)U(\tau,s)x_0\right] = -U(t,\tau)A(\tau)U(\tau,s)x_0 + U(t,\tau)A(\tau)U(\tau,s)x_0 = 0 \\ \Rightarrow \qquad &U(t,\tau)U(\tau,s)x_0 = U(t,t)U(t,s)x_0 = U(t,s)x_0.\end{aligned}$$

Since the $U(t,s)$ are bounded and $\mathcal{D}(A)$ is dense in E, the assertion follows.

□

4.3 Heat kernel asymptotic

In this section the heat kernel $e^{-tT}(x,y)$ of a Laplace-like operator

$$T := -\frac{\partial^2}{\partial x^2} + \frac{1}{x^2}A(x)$$

is expanded into an asymptotic series. This exposition follows [Brü88], Chapter 2.

Definition 4.3.1. Let $H = H_0 \supset H_1 \dots$ be a sequence of Hilbert spaces. We denote by

$$(0,\infty) \ni x \mapsto A(x) \in \mathcal{L}\left(H_j, H_{j-1}\right) \text{ for all } j \in \mathbb{N}$$

a smooth family of self-adjoint operators satisfying the properties:

1. There is a $c \in \mathbb{R}$, such that $A(x) \geqslant -c+1$ for all $x \in (0,\infty)$.
2. There is a $p_0 > 0$, such that $(A(x)+c)^{-1}$ is in the von Neumann-Schatten class $C_{p_0}(H)$.
3. For all $x \in (0,\infty)$ and $k \in \mathbb{N}\cup\{0\}$ there is a $C_k > 0$, such that

$$\left\| A^{(k)}(x)(A(x)+c)^{-1} \right\|_{\mathcal{L}(H)} \leqslant C_k.$$

4. $A(x)A(y) = A(y)A(x)$ for all $x,\ y \in (0,\infty)$.

We define

$$T := -\frac{\partial^2}{\partial x^2} + \frac{1}{x^2}A(x).$$

For the asymptotic expansion of the heat kernel of T we make the ansatz

$$e^{-tT}(x,y) \sim (4\pi t)^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{4t}} \sum_{j=0}^{\infty} t^j U_j(x,y) e^{-ty^{-2}A(y)}.$$

First, we compute the t-derivative

$$\begin{aligned}
\frac{\partial}{\partial t}\left(t^{-\frac{1}{2}+j} e^{-\frac{(x-y)^2}{4t}} e^{-ty^{-2}A(y)}\right) &= \left(j-\frac{1}{2}\right) t^{-\frac{3}{2}+j} e^{-\frac{(x-y)^2}{4t}} e^{-ty^{-2}A(y)} \\
&\quad + t^{-\frac{1}{2}+j} \frac{(x-y)^2}{4t^2} e^{-\frac{(x-y)^2}{4t}} e^{-ty^{-2}A(y)} - t^{-\frac{1}{2}+j} e^{-\frac{(x-y)^2}{4t}} y^{-2}A(y) e^{-ty^{-2}A(y)} \\
&= t^{-\frac{1}{2}+j} e^{-\frac{(x-y)^2}{4t}} \left[-y^{-2}A(y) + \frac{1}{t}\left(j-\frac{1}{2}\right) + \frac{1}{t^2}\frac{(x-y)^2}{4}\right] e^{-ty^{-2}A(y)}
\end{aligned}$$

and proceed with the x-derivative

$$\begin{aligned}
&-\frac{\partial^2}{\partial x^2}\left(e^{-\frac{(x-y)^2}{4t}} U_j(x,y)\right) = -\frac{\partial}{\partial x}\left(-\frac{x-y}{2t} e^{-\frac{(x-y)^2}{4t}} U_j(x,y) + e^{-\frac{(x-y)^2}{4t}} \frac{\partial U_j}{\partial x}(x,y)\right) \\
&= \frac{1}{2t} e^{-\frac{(x-y)^2}{4t}} U_j(x,y) - \frac{(x-y)^2}{4t^2} e^{-\frac{(x-y)^2}{4t}} U_j(x,y) + \frac{x-y}{2t} e^{-\frac{(x-y)^2}{4t}} \frac{\partial U_j}{\partial x}(x,y) \\
&\quad + \frac{x-y}{2t} e^{-\frac{(x-y)^2}{4t}} \frac{\partial U_j}{\partial x}(x,y) - e^{-\frac{(x-y)^2}{4t}} \frac{\partial^2 U_j}{\partial x^2}(x,y) \\
&= e^{-\frac{(x-y)^2}{4t}} \left\{-\frac{\partial^2 U_j}{\partial x^2}(x,y) + \frac{1}{t}\left[\frac{1}{2}U_j(x,y) + (x-y)\frac{\partial U_j}{\partial x}(x,y)\right] - \frac{1}{t^2}\frac{(x-y)^2}{4} U_j(x,y)\right\}.
\end{aligned}$$

Multiplying by $(4\pi)^{-\frac{1}{2}} U_j(x,y)$ and $(4\pi t)^{-\frac{1}{2}} t^j e^{-ty^{-2}A(y)}$, respectively, yields

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + x^{-2}A(x)\right)\left[(4\pi t)^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{4t}} \sum_{j=0}^{N} t^j U_j(x,y) e^{-ty^{-2}A(y)}\right] \\
&\quad = (4\pi t)^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{4t}} \sum_{j=0}^{N} \Bigg\{ t^j \left[-\frac{\partial^2 U_j}{\partial x^2}(x,y) + \left(x^{-2}A(x) - y^{-2}A(y)\right) U_j(x,y)\right] \\
&\qquad + t^{j-1}\left[jU_j(x,y) + (x-y)\frac{\partial U_j}{\partial x}(x,y)\right]\Bigg\} e^{-ty^{-2}A(y)} \\
&\quad = (4\pi t)^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{4t}} \Bigg[t^{-1}(x-y)\frac{\partial U_0}{\partial x}(x,y) \\
&\qquad + \sum_{j=0}^{N-1} t^j \Bigg\{ -\frac{\partial^2 U_j}{\partial x^2}(x,y) + \left(x^{-2}A(x) - y^{-2}A(y)\right) U_j(x,y) \\
&\qquad\qquad + (j+1)U_{j+1}(x,y) + (x-y)\frac{\partial U_{j+1}}{\partial x}(x,y) \Bigg\} \\
&\qquad + t^N \left\{-\frac{\partial^2 U_N}{\partial x^2}(x,y) + \left(x^{-2}A(x) - y^{-2}A(y)\right) U_N(x,y)\right\} \Bigg] e^{-ty^{-2}A(y)}.
\end{aligned}$$

Thus, the U_j have to satisfy the recursion equations

$$\begin{aligned} \frac{\partial}{\partial x} U_0(x,y) &= 0 \qquad\qquad \text{and} \\ (x-y)\frac{\partial U_{j+1}}{\partial x}(x,y) + (j+1)U_{j+1}(x,y) &= \frac{\partial^2 U_j}{\partial x^2}(x,y) - \left(x^{-2}A(x) - y^{-2}A(y)\right) U_j(x,y) \\ &=: R_j(x,y). \end{aligned} \tag{4.9}$$

Lemma 4.3.2. [12] *The recursively defined functions*

$$\begin{aligned} U_0(x,y) &= \mathbb{1} \\ U_{j+1}(x,y) &= \int_y^x \frac{(z-y)^j}{(x-y)^{j+1}} R_j(z,y)dz = \int_0^1 s^j R_j(y+s(x-y),y)ds, \quad j \in \mathbb{N}\cup\{0\} \end{aligned}$$

have the property that $U_j \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+, \mathcal{L}(H_{k+j}, H_k))$ for all $k \in \mathbb{N}\cup\{0\}$. Moreover, they satisfy the equations (4.9) *and*

$$U_j(x,y)A(z)e = A(z)U_j(x,y)e, \qquad \forall e \in H_{j+1}.$$

Proof. We have to prove three assertions.

1. Regularity: We prove that $U_j \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+, \mathcal{L}(H_{k+j}, H_k))$ for all $k \in \mathbb{N}\cup\{0\}$ by induction over $j \in \mathbb{N}\cup\{0\}$. The initial step $j = 0$ is clear. Let us assume the assertion is proved for a $j \in \mathbb{N}\cup\{0\}$. Since, by assumption, $U_j \in \mathbb{C}^\infty(\mathbb{R}_+ \times \mathbb{R}_+, \mathcal{L}(H_{k+1+j}, H_{k+1}))$ for all $k \in \mathbb{N}\cup\{-1,0\}$, we find

$$\begin{aligned} R_j(x,y) = \frac{\partial^2 U_j}{\partial x^2}(x,y) - \big(x^{-2}A(x) - y^{-2}A(y)\big)U_j(x,y) \\ \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+, \mathcal{L}(H_{k+j+1}, H_k)) \end{aligned}$$

   for all $k \in \mathbb{N}\cup\{0\}$ and thus, $U_{j+1} \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+, \mathcal{L}(H_{k+(j+1)}, H_k))$ for all $k \in \mathbb{N}\cup\{0\}$.

2. U_j satisfies (4.9):

$$\begin{aligned} \left[(x-y)\frac{\partial}{\partial x} + (j+1)\right] U_{j+1}(x,y) &= \left[(x-y)\frac{\partial}{\partial x} + (j+1)\right] \int_y^x \frac{(z-y)^j}{(x-y)^{j+1}} R_j(z,y)dz \\ &= (x-y)\left[\frac{1}{(x-y)}R_j(x,y) - (j+1)\int_y^x \frac{(z-y)^j}{(x-y)^{j+2}} R_j(z,y)dz\right] \\ &\quad + (j+1)\int_y^x \frac{(z-y)^j}{(x-y)^{j+1}} R_j(z,y)dz = R_j(x,y). \end{aligned}$$

[12] This is Lemma 2.1 in [Brü88]. We added some details to the proof.

3. Commutativity: We prove the assertion by induction over $j \in \mathbb{N} \cup \{0\}$. The initial step $j = 0$ is clear. Let us assume the assertion is proved for a $j \in \mathbb{N} \cup \{0\}$.

$$
\begin{aligned}
A(z)U_{j+1}(x,y) &= \int_y^x \frac{(u-y)^j}{(x-y)^{j+1}} A(z) R_j(u,y) du \\
&\overset{[A(x),A(y)]=0}{=} \int_y^x \frac{(u-y)^j}{(x-y)^{j+1}} \left[\frac{\partial^2}{\partial u^2} A(z) U_j(u,y) - \left(u^{-2}A(u) - y^{-2}A(y)\right) A(z) U_j(u,y) \right] du \\
&\overset{[A,U_j]=0}{=} \int_y^x \frac{(u-y)^j}{(x-y)^{j+1}} R_j(u,y) A(z) du = U_{j+1}(x,y) A(z).
\end{aligned}
$$

□

In a manner of speaking, the following lemma is the heart of this section. Without the slight degree drop – caused by cancellation – most of the estimates necessary for the asymptotic expansion would not work.

Lemma 4.3.3. [13] *The operators*

$$
U_{jk}(y) := \frac{1}{k!} \frac{\partial^k U_j}{\partial x^k}(y,y), \quad j,\ k \in \mathbb{N} \cup \{0\}
$$

satisfy the recursion formulas

$$
\begin{aligned}
U_{00}(y) &\equiv \mathbb{1},\ U_{0k}(y) \equiv 0, \quad k \in \mathbb{N}, \\
U_{j+1,k}(y) &= \frac{1}{j+k+1} \Bigg((k+1)(k+2) U_{j,k+2}(y) \\
&\qquad - \sum_{l=0}^{k-1} U_{jl}(y) \sum_{m=0}^{k-l} (-1)^m \frac{m+1}{(k-l-m)!} y^{-m-2} A^{(k-l-m)}(y) \Bigg).
\end{aligned}
$$

$U_{jk}(y)$ is a polynomial in the variables $A^{(i)}(y),\ i \in \mathbb{N} \cup \{0\}$ of degree

$$
d_{jk} \leqslant \min \left\{ \left\lfloor \frac{2}{3} j + \frac{1}{3} k \right\rfloor, j \right\}
$$

with coefficients in $\mathbb{R}[y^{-1}]$.

[13] This is Lemma 2.2 in [Brü88]. We rendered more precisely what is meant by universal polynomial and added some details to the proof.

Proof. Let $j \in \mathbb{N} \cup \{0\}$.

$$
\begin{aligned}
U_{j+1,k}(y) &= \frac{1}{k!}\frac{\partial^k U_{j+1}}{\partial x^k}(y,y) = \frac{1}{k!}\left.\frac{\partial^k}{\partial x^k}\right|_{x=y} \int_0^1 s^j R_j(y+s(x-y),y)ds \\
&= \frac{1}{k!}\int_0^1 s^j \frac{\partial^k R_j}{\partial x^k}(y,y)s^k ds = \frac{1}{j+k+1}\frac{1}{k!}\frac{\partial^k R_j}{\partial x^k}(y,y) \\
&= \frac{1}{j+k+1}\frac{1}{k!}\left.\frac{\partial^k}{\partial x^k}\right|_{x=y}\left[\frac{\partial^2 U_j}{\partial x^2}(x,y) - \left(x^{-2}A(x) - y^{-2}A(y)\right)U_j(x,y)\right] \\
&= \frac{1}{j+k+1}\left[\frac{1}{k!}\frac{\partial^{k+2}U_j}{\partial x^{k+2}}(y,y) - \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}\frac{\partial^l U_j}{\partial x^l}(y,y)\left.\frac{\partial^{k-l}}{\partial x^{k-l}}\right|_{x=y}\left(x^{-2}A(x) - y^{-2}A(y)\right)\right] \\
&= \frac{1}{j+k+1}\Bigg[(k+2)(k+1)U_{j,k+2}(y) \\
&\qquad - \sum_{l=0}^{k-1}\frac{1}{(k-l)!}U_{jl}(y)\sum_{m=0}^{k-l}\binom{k-l}{m}(-1)^m(m+1)!y^{-2-m}A^{(k-l-m)}(y)\Bigg] \\
&= \frac{1}{j+k+1}\Bigg[(k+2)(k+1)U_{j,k+2}(y) \\
&\qquad - \sum_{l=0}^{k-1}U_{jl}(y)\sum_{m=0}^{k-l}(-1)^m\frac{m+1}{(k-l-m)!}y^{-2-m}A^{(k-l-m)}(y)\Bigg]
\end{aligned}
$$

Next, we prove the degree formula by induction over j. $d_{0k} = 0$ for $k \in \mathbb{N} \cup \{0\}$, and we assume that the formula is proved for a $j \in \mathbb{N} \cup \{0\}$.

$$
\begin{aligned}
d_{j+1,k} &\leqslant \max\left\{d_{j,k+2}, \max_{0 \leqslant l \leqslant k-1}\{d_{jl}\} + 1\right\} \\
&\leqslant \max\left\{\min\left\{\left\lfloor\frac{2}{3}j + \frac{1}{3}(k+2)\right\rfloor, j\right\}, \max_{0\leqslant l\leqslant k-1}\left\{\min\left\{\left\lfloor\frac{2}{3}j + \frac{1}{3}l\right\rfloor, j\right\}\right\} + 1\right\} \\
&= \max\left\{\min\left\{\left\lfloor\frac{2}{3}(j+1) + \frac{1}{3}k\right\rfloor, j\right\}, \min\left\{\left\lfloor\frac{2}{3}j + \frac{1}{3}(k-1)\right\rfloor, j\right\} + 1\right\} \\
&= \max\left\{\min\left\{\left\lfloor\frac{2}{3}(j+1) + \frac{1}{3}k\right\rfloor, j\right\}, \min\left\{\left\lfloor\frac{2}{3}(j+1) + \frac{1}{3}k\right\rfloor, j+1\right\}\right\} \\
&= \min\left\{\left\lfloor\frac{2}{3}(j+1) + \frac{1}{3}k\right\rfloor, j+1\right\}.
\end{aligned}
$$
□

The N^{th}-approximation of the heat kernel is defined by

$$
H_t^N u(x) := \int_0^\infty H_t^N(x,y)u(y)dy, \quad u \in L^2(\mathbb{R}_+, H),
$$

where

$$
H_t^N(x,y) := (4\pi t)^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{4t}} \sum_{j=0}^{N} t^j U_j(x,y) e^{-ty^{-2}A(y)}.
$$

Lemma 4.3.4. [14] *Let $\varphi, \psi \in C_0^\infty(\mathbb{R}_+, [0,1])$, such that $\psi \equiv 1$ in a neighbourhood of* $\operatorname{supp}\varphi$*. Then*

$$\lim_{t\to 0} \left\|\psi H_t^N \varphi u - \varphi u\right\|_{L^2(\mathbb{R}_+,H)} = 0 \qquad \text{for } u \in L^2(\mathbb{R}_+, H).$$

Proof. During the proof we will need the following estimate several times

$$(4\pi t)^{-\frac{1}{2}} \int_0^\infty e^{-\frac{(x-y)^2}{4t}} dy \leqslant \pi^{-\frac{1}{2}} \int_{-\infty}^\infty e^{-z^2} dz = \pi^{-\frac{1}{2}} 2\sqrt{\frac{\pi}{4}} = 1.$$

Inserting $u \in L^2(\mathbb{R}_+, H)$, implies

$$\begin{aligned}
&\left\| \int_0^\infty (4\pi t)^{-\frac{1}{2}} e^{-\frac{(\cdot - y)^2}{4t}} u(y) dy \right\|^2_{L^2(\mathbb{R}_+,H)} \\
&\qquad = \int_0^\infty \left(\int_0^\infty (4\pi t)^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{4t}} \left\|u(y)\right\|_H dy \right)^2 dx \\
&\qquad \overset{CS}{\leqslant} \int_0^\infty \underbrace{\int_0^\infty (4\pi t)^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{4t}} dy}_{\leqslant 1} \int_0^\infty (4\pi t)^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{4t}} \left\|u(y)\right\|_H^2 dy dx \\
&\qquad \overset{Fubini}{\leqslant} \int_0^\infty \underbrace{\int_0^\infty (4\pi t)^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{4t}} dx}_{\leqslant 1} \left\|u(y)\right\|_H^2 dy \leqslant \left\|u\right\|^2_{L^2(\mathbb{R}_+,H)}.
\end{aligned} \tag{4.10}$$

For $u \in L^2(\mathbb{R}_+, H)$ and $t \in (0,1]$ we find (by calculating as in the norm estimate for $\widetilde{H}_t^N$ below)

$$\left\|\psi H_t^N \varphi u\right\|_{L^2(\mathbb{R}_+,H)} \leqslant C_N \left\| \int_0^\infty (4\pi t)^{-\frac{1}{2}} e^{-\frac{(\cdot - y)^2}{4t}} u(y) dy \right\|_{L^2(\mathbb{R}_+,H)} \leqslant C \left\|u\right\|_{L^2(\mathbb{R}_+,H)}$$

and thus,

$$\left\|\psi H_t^N \varphi\right\|_{\mathcal{L}(L^2(\mathbb{R}_+,H))} \leqslant C,$$

i.e. the operator $\psi H_t^N \varphi$ is uniformly bounded for $t \in (0,1]$. Therefore, it suffices to prove the assertion of the lemma for $u \in C_0(\mathbb{R}_+, H)$.

Let

$$\begin{aligned}
\widetilde{H}_t^N(x,y) &:= \psi(x) H_t^N(x,y) \varphi(y) - (4\pi t)^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{4t}} \psi(x) e^{-ty^{-2}A(y)} \varphi(y) \\
\widetilde{H}_t^N u(x) &:= \int_0^\infty \widetilde{H}_t^N(x,y) u(y) dy, \qquad u \in L^2(\mathbb{R}_+, H)
\end{aligned}$$

[14] This is Lemma 2.3 in [Brü88]. We just added the missing φ to the statement and more details to the proof.

and estimate its norm with the help of Lemma 4.3.3

$$\begin{aligned}
\left\|\widetilde{H}_t^N(x,y)\right\|_{\mathcal{L}(H)} &= \left\|(4\pi t)^{-\frac{1}{2}}e^{-\frac{(x-y)^2}{4t}}\sum_{j=1}^N t^j\psi(x)U_j(x,y)\varphi(y)e^{-ty^{-2}A(y)}\right\|_{\mathcal{L}(H)} \\
&\overset{Taylor}{\leqslant} (4\pi t)^{-\frac{1}{2}}e^{-\frac{(x-y)^2}{4t}}\sum_{j=1}^N t^{\lceil \frac{j}{3}\rceil}\psi(x)y^{\lceil \frac{4}{3}j\rceil}\left\|\left[ty^{-2}\right]^{\lfloor \frac{2}{3}j\rfloor}U_{j0}(y)e^{-ty^{-2}A(y)}\right\|_{\mathcal{L}(H)}\varphi(y) \\
&\quad + (4\pi t)^{-\frac{1}{2}}e^{-\frac{(x-y)^2}{4t}}\sum_{j=1}^N \psi(x)(x-y)\int_0^1\left\|t^j\frac{\partial U_j}{\partial x}(y+u(x-y),y)e^{-ty^{-2}A(y)}\right\|_{\mathcal{L}(H)}du\ \varphi(y) \\
&\overset{4.3.3}{\leqslant} C(4\pi t)^{-\frac{1}{2}}\left[\sum_{j=1}^N e^{-\frac{(x-y)^2}{4t}}t^{\lceil \frac{j}{3}\rceil}+e^{-\frac{(x-y)^2}{8t}}t^{\frac{1}{2}}\sqrt{8}e^{-\frac{(x-y)^2}{8t}}\frac{(x-y)}{\sqrt{8t}}\right]\leqslant Ct^{\frac{1}{2}}(4\pi t)^{-\frac{1}{2}}e^{-\frac{(x-y)^2}{8t}}
\end{aligned}$$

since $\operatorname{supp}\psi$ and $\operatorname{supp}\varphi$ are compact.

$$\begin{aligned}
\left\|\widetilde{H}_t^N u\right\|_{L^2(\mathbb{R}_+,H)} &= \left\|\int_0^\infty \widetilde{H}_t^N(\cdot,y)u(y)dy\right\|_{L^2(\mathbb{R}_+,H)} \\
&\leqslant Ct^{\frac{1}{2}}\left\|\int_0^\infty (4\pi t)^{-\frac{1}{2}}e^{-\frac{(\cdot-y)^2}{8t}}u(y)dy\right\|_{L^2(\mathbb{R}_+,H)} \\
&\overset{(4.10)}{\leqslant} Ct^{\frac{1}{2}}\left\|u\right\|_{L^2(\mathbb{R}_+,H)} =: C_u t^{\frac{1}{2}}
\end{aligned}$$

Let $u \in C_0(\mathbb{R}_+,H)$. Then

$$\begin{aligned}
&\psi H_t^N\varphi u(x)-\varphi u(x) = \int_0^\infty \psi(x)H_t^N(x,y)\varphi(y)u(y)dy-\varphi(x)u(x) \\
&= \int_0^\infty (4\pi t)^{-\frac{1}{2}}e^{-\frac{(x-y)^2}{4t}}\psi(x)e^{-ty^{-2}A(y)}\varphi(y)u(y)dy-\varphi(x)u(x)+\left(\widetilde{H}_t^N u\right)(x) \\
&\overset{\operatorname{supp}\varphi\subseteq(0,\infty)}{=} \int_{-\infty}^\infty (4\pi t)^{-\frac{1}{2}}e^{-\frac{(x-y)^2}{4t}}\psi(x)\left[e^{-ty^{-2}A(y)}\varphi(y)u(y)-\varphi(x)u(x)\right]dy+O_{L^2}(t^{\frac{1}{2}}) \\
&= \int_0^\infty (4\pi t)^{-\frac{1}{2}}e^{-\frac{(x-y)^2}{4t}}\psi(x)\left[e^{-ty^{-2}A(y)}-e^{-tx^{-2}A(x)}\right]\varphi(y)u(y)dy \\
&\quad + \int_{-\infty}^\infty (4\pi t)^{-\frac{1}{2}}e^{-\frac{(x-y)^2}{4t}}\psi(x)e^{-tx^{-2}A(x)}\left[\varphi(y)u(y)-\varphi(x)u(x)\right]dy \\
&\quad + \varphi(x)\left[e^{-tx^{-2}A(x)}-\mathbb{1}\right]u(x)\underbrace{\int_{-\infty}^\infty (4\pi t)^{-\frac{1}{2}}e^{-\frac{(x-y)^2}{4t}}dy}_{=1}+O_{L^2}(t^{\frac{1}{2}})
\end{aligned}$$

where we have used that $\psi\varphi=\varphi$. We estimate the L^2-norms of all three terms separately:

1. The estimate

$$\begin{aligned}
&e^{-\frac{(x-y)^2}{4t}}\psi(x)\left\|e^{-tx^{-2}A(x)}-e^{-ty^{-2}A(y)}\right\|\varphi(y)\\
&\qquad\qquad= e^{-\frac{(x-y)^2}{4t}}\psi(x)\left\|\int_y^x \frac{\partial}{\partial z}e^{-tz^{-2}A(z)}dz\right\|\varphi(y)\\
&\qquad\qquad\leqslant e^{-\frac{(x-y)^2}{4t}}(x-y)\sup_{z\in\operatorname{supp}\psi}\left\|t(2z^{-3}A(z)-z^{-2}A'(z))e^{-tz^{-2}A(z)}\right\|\\
&\qquad\qquad\leqslant C(x-y)e^{-\frac{(x-y)^2}{4t}}\leqslant C\sqrt{t}e^{-\frac{(x-y)^2}{8t}}
\end{aligned}$$

   implies

$$\begin{aligned}
&\left\|\int_0^\infty (4\pi t)^{-\frac{1}{2}}e^{-\frac{(\cdot-y)^2}{4t}}\psi(\cdot)\left[e^{-ty^{-2}A(y)}-e^{-t\cdot^{-2}A(\cdot)}\right]\varphi(y)u(y)dy\right\|_{L^2(\mathbb{R}_+,H)}\\
&\qquad\qquad\leqslant C\sqrt{t}\left\|\int_0^\infty (4\pi t)^{-\frac{1}{2}}e^{-\frac{(\cdot-y)^2}{8t}}u(y)dy\right\|_{L^2(\mathbb{R}_+,H)}\overset{(4.10)}{\leqslant} C_u\sqrt{t}.
\end{aligned}$$

2. Given an $\varepsilon>0$, choose $\delta>0$, such that

$$\|\varphi u(x)-\varphi u(y)\|_H\leqslant\frac{\varepsilon}{\sqrt{2\operatorname{vol}(\operatorname{supp}\psi)}},\qquad\forall\, x,y\in\mathbb{R}_+\text{ with }\|x-y\|\leqslant\delta.$$

   This is possible since φu is continuous and has compact support and thus is uniformly continuous. Next, choose a $t>0$ small enough, such that

$$2e^{-\frac{\delta^2}{8t}}\|u\|_{L^2(\mathbb{R}_+,H)}\leqslant\frac{\varepsilon}{\sqrt{2}}.$$

This gives

$$\begin{aligned}
&\left\| \int_0^\infty (4\pi t)^{-\frac{1}{2}} e^{-\frac{(\cdot-y)^2}{4t}} \psi(\cdot) e^{-t\cdot^{-2}A(\cdot)} \left[\varphi u(y) - \varphi u(\cdot)\right] dy \right\|^2_{L^2(\mathbb{R}_+,H)} \\
&\overset{\text{as in (4.10)}}{\leqslant} \int_0^\infty \psi(x) \int_0^\infty (4\pi t)^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{4t}} \|\varphi u(y) - \varphi u(x)\|_H^2 \, dydx \\
&\leqslant \int_0^\infty \psi(x) \int_{x-\delta}^{x+\delta} (4\pi t)^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{4t}} \|\varphi u(y) - \varphi u(x)\|_H^2 \, dydx \\
&\qquad + 2\int_0^\infty \int_{\mathbb{R}\setminus(y-\delta,y+\delta)} (4\pi t)^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{4t}} dx \, \|\varphi u(y)\|_H^2 \, dy \\
&\qquad + 2\int_0^\infty \int_{\mathbb{R}\setminus(x-\delta,x+\delta)} (4\pi t)^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{4t}} dy \, \|\varphi u(x)\|_H^2 \, dx \\
&\leqslant \frac{\varepsilon^2}{2\operatorname{vol}(\operatorname{supp}\psi)} \int_0^\infty \psi(x) \underbrace{\int_{\mathbb{R}} (4\pi t)^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{4t}} dy}_{=1} dx \\
&\qquad + 2e^{-\frac{\delta^2}{4t}} \left[\int_0^\infty \|\varphi u(y)\|_H^2 \, dy + \int_0^\infty \|\varphi u(x)\|_H^2 \, dx \right] \\
&\leqslant \frac{\varepsilon^2}{2} + 4e^{-\frac{\delta^2}{4t}} \|u\|^2_{L^2(\mathbb{R}_+,H)} \leqslant \varepsilon^2.
\end{aligned}$$

3. Due to the properties of $A(x)$ and the spectral theorem:

$$\left\| e^{-tx^{-2}A(x)} - \mathbb{1} \right\|_{\mathcal{L}(H)} \leqslant \left| e^{-tx^{-2}(1-c)} - 1 \right| \xrightarrow{t\to 0} 0.$$

Since $\operatorname{supp}\varphi$ is compact, this implies

$$\left\| \varphi(\cdot) \left[e^{-t\cdot^{-2}A(\cdot)} - \mathbb{1} \right] u(\cdot) \right\|_{L^2(\mathbb{R}_+,H)} \xrightarrow{t\to 0} 0.$$

Taking all three estimates together, proves the assertion

$$\left\| \psi H_t^N \varphi u - \varphi u \right\|_{L^2(\mathbb{R}_+,H)} \xrightarrow{t\to 0} 0.$$

□

Theorem 4.3.5. [15] *Let T be a semibounded self-adjoint extension in $L^2((0,\infty),H)$ of the operator $-\partial_x^2 + x^{-2}A(x)$, where A is defined as in Definition 4.3.1. The heat operator e^{-tT}*

[15] This is Theorem 2.1 in [Brü88]. In the proof in [Brü88], Lemma 2.4 in the same article and Lemma 4.1 in the appendix of [BS87] are applied. The proof of Lemma 2.4 does not work. It uses estimates of the kind $\left\|\partial_y R_t^N(x,y)\right\|_{C_p(H)} \leqslant Ct^{\nu_N}$ (Formula 2.15 on page 81) which cannot be achieved since the t-exponents cancel out in this case. Fortunately, the assertion can be proved by estimating the norm of the kernels' difference directly.

has an operator kernel $e^{-tT}(x,y) \in C_1(H)$, for $x,y,t > 0$. For $N \in \mathbb{N}$ large enough exists a constant C_N, such that

$$\left\| e^{-tT}(x,y) - (4\pi t)^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{4t}} \sum_{j=0}^{N} t^j U_j(x,y) e^{-ty^{-2}A(y)} \right\|_{\mathrm{tr}} \leqslant C_N t^{\nu_N}$$

with $\lim_{N\to\infty} \nu_N = \infty$. The constants C_N can be chosen uniformly for (x,y) in a compact subset of $\mathbb{R}_+ \times \mathbb{R}_+$.

Proof. For a compact subset $K \subset \mathbb{R}_+ \times \mathbb{R}_+$ choose $\psi, \varphi \in C_0^\infty(\mathbb{R}_+, [0,1])$, such that $\psi \times \varphi|_K \equiv 1$ and $\psi|_{\operatorname{supp}\varphi} \equiv 1$. Applying $\frac{\partial}{\partial t} + T$ to the N^{th} approximation kernel, gives a rest term:

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} + T\right) \psi(x) H_t^N(x,y)\varphi(y) \\
&= \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + x^{-2}A(x)\right) \psi(x) \left[(4\pi t)^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{4t}} \sum_{j=0}^{N} t^j U_j(x,y) e^{-ty^{-2}A(y)} \right] \varphi(y) \\
&= -\psi''(x) H_t^N(x,y)\varphi(y) - 2\psi'(x) \frac{\partial H_t^N}{\partial x}(x,y)\varphi(y) + \psi(x) \left(\frac{\partial}{\partial t} + T\right) H_t^N(x,y)\varphi(y) \\
&= -\psi''(x) H_t^N(x,y)\varphi(y) - 2\psi'(x) \frac{\partial H_t^N}{\partial x}(x,y)\varphi(y) \\
&+ \psi(x)(4\pi t)^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{4t}} t^N \left[-\frac{\partial^2 U_N}{\partial x^2}(x,y) + (x^{-2}A(x) - y^{-2}A(y)) U_N(x,y) \right] e^{-ty^{-2}A(y)} \varphi(y) \\
&\overset{(4.9)}{=} -\psi''(x) H_t^N(x,y)\varphi(y) - 2\psi'(x) \frac{\partial H_t^N}{\partial x}(x,y)\varphi(y) \\
&\quad - \psi(x)(4\pi t)^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{4t}} t^N \left[(x-y) \frac{\partial U_{N+1}}{\partial x}(x,y) + (N+1) U_{N+1}(x,y) \right] e^{-ty^{-2}A(y)} \varphi(y) \\
&=: R_t^N(x,y).
\end{aligned}$$

Integrating the kernels, gives

$$\left(\frac{\partial}{\partial t} + T\right) \left(\psi H_t^N \varphi u\right)(x) = \int_0^\infty R_t^N(x,y) u(y) dy =: \left(R_t^N u\right)(x).$$

Since T is semibounded and self-adjoint, the spectral theorem implies that e^{-tT} exists and has the usual properties. Thus,

$$\begin{aligned}
\frac{\partial}{\partial s}\left(e^{-(t-s)T} \psi H_s^N \varphi u\right) &= T e^{-(t-s)T} \psi H_s^N \varphi u + e^{-(t-s)T} (-T\psi H_s^N \varphi u + R_s^N u) \\
&= e^{-(t-s)T} R_s^N u.
\end{aligned}$$

Let $\chi \in C_0^\infty(\mathbb{R}_+, [0,1])$, such that $\chi\psi = \psi$. Applying Lemma 4.3.4 and the analogous statement for the heat kernel, gives

$$\begin{aligned}
&\left\| \int_0^t \chi e^{-(t-s)T} R_s^N u ds - \psi H_t^N \varphi u + \chi e^{-tT} \varphi u \right\|_{L^2(\mathbb{R}_+,H)} \\
&\quad = \left\| \int_0^t \frac{\partial}{\partial s} \left(\chi e^{-(t-s)T} \psi H_s^N \varphi u \right) ds - \psi H_t^N \varphi u + \chi e^{-tT} \varphi u \right\|_{L^2(\mathbb{R}_+,H)} \\
&\quad = \left\| \lim_{s\to t} \chi e^{-(t-s)T} \psi H_s^N \varphi u - \psi H_t^N \varphi u - \left(\lim_{s\to 0} \chi e^{-(t-s)T} \psi H_s^N \varphi u - \chi e^{-tT} \varphi u \right) \right\|_{L^2(\mathbb{R}_+,H)} \\
&\quad \leqslant \lim_{s\to t} \left\| \chi e^{-(t-s)T} \chi \right\| \left\| \psi H_s^N \varphi u - \psi H_t^N \varphi u \right\|_{L^2(\mathbb{R}_+,H)} \\
&\qquad + \lim_{s\to t} \left\| \chi e^{-(t-s)T} \psi H_t^N \varphi u - \psi H_t^N \varphi u \right\|_{L^2(\mathbb{R}_+,H)} \\
&\qquad + \lim_{s\to 0} \left\| \chi e^{-(t-s)T} \chi \right\| \left\| \psi H_s^N \varphi u - \varphi u \right\|_{L^2(\mathbb{R}_+,H)} \\
&\qquad + \lim_{s\to 0} \left\| \chi e^{-(t-s)T} \varphi u - \chi e^{-tT} \varphi u \right\|_{L^2(\mathbb{R}_+,H)} = 0,
\end{aligned}$$

i.e.

$$\psi H_t^N \varphi u - \chi e^{-tT} \varphi u = \int_0^t \chi e^{-(t-s)T} R_s^N u ds$$

as operators in $\mathcal{L}(L^2(\mathbb{R}_+, H))$ and

$$\psi(x) H_t^N(x,y) \varphi(y) - \chi(x) e^{-tT}(x,y) \varphi(y) = \int_0^t \int_0^\infty \chi(x) e^{-(t-s)T}(x,z) R_s^N(z,y) dz ds.$$

$$\Rightarrow \left\| \psi(x) H_t^N(x,y) \varphi(y) - \chi(x) e^{-tT}(x,y) \varphi(y) \right\|_{\mathrm{tr}} \leqslant C \chi(x) \int_0^t \int_0^\infty \left\| R_s^N(z,y) \right\|_{\mathrm{tr}} dz ds. \quad (4.11)$$

Therefore, we have to compute an estimate for $\left\| R_s^N(x,y) \right\|_{\mathrm{tr}}$. Before the norm is estimated, it is necessary to do some auxiliary calculations. To a greater or lesser extent they are applications of Lemma 4.3.3 and the fact that for $a > 0$

$$x^k e^{-ax} \leqslant C, \text{ for all } x \in \mathbb{R}.$$

By assumption $A(y) + c \geqslant 1$ and it exists a p_0, such that $(A(y)+c)^{-1} \in C_{p_0}(H)$ for all $y > 0$. For $y \in \operatorname{supp} \varphi$ it follows that

$$\begin{aligned}
\left\| (A(y)+c)^l \, e^{-ty^{-2} A(y)} \right\|_{\mathrm{tr}} &= \sum_{\lambda \in \operatorname{spec} A(y)} (\lambda + c)^l \, e^{-ty^{-2}\lambda} \\
&\leqslant C \sum_{\lambda \in \operatorname{spec} A(y)} (\lambda + c)^{-p_0} t^{-p_0 - l} \qquad \text{since } y \leqslant \sup(\operatorname{supp} \varphi) \\
&\leqslant C t^{-p_0 - l} \left\| (A(y)+c)^{-1} \right\|_{C_{p_0}(H)}^{p_0} \leqslant C t^{-p_0 - l}. \quad (4.12)
\end{aligned}$$

Let $a \in \mathbb{N}$, $a \leqslant j$, $x \in \operatorname{supp}\psi$ and $y \in \operatorname{supp}\varphi$. Due to the Taylor expansion, there is a $\vartheta \in [0,1]$, such that

$$\begin{aligned}
&\left\| t^j (x-y)^a \frac{\partial^a U_j}{\partial x^a}(x,y) e^{-ty^{-2}A(y)} \right\|_{\mathrm{tr}} \\
&\overset{\text{Taylor}}{\leqslant} \sum_{k=0}^{j-a-1} \frac{|x-y|^{a+k}}{k!}(a+k)! \left\| t^j U_{j,a+k}(y) e^{-ty^{-2}A(y)} \right\|_{\mathrm{tr}} \\
&\qquad + \frac{|x-y|^j}{(j-a)!} \left\| t^j \frac{\partial^j U_j}{\partial x^j}(x+\vartheta(y-x),y) e^{-ty^{-2}A(y)} \right\|_{\mathrm{tr}} \\
&\leqslant \sum_{k=0}^{j-a-1} \frac{|x-y|^{a+k}}{k!}(a+k)! \left\| U_{j,a+k}(y)(A(y)+c)^{-\lfloor \frac{2}{3}j + \frac{a+k}{3} \rfloor} \right\|_{\mathcal{L}(H)} \\
&\qquad\quad \cdot \left\| [t(A(y)+c)]^{\lfloor \frac{2}{3}j + \frac{a+k}{3} \rfloor} e^{-ty^{-2}A(y)} \right\|_{\mathrm{tr}} t^{\lceil \frac{j}{3} - \frac{a+k}{3} \rceil} \\
&\quad + \frac{|x-y|^j}{(j-a)!} \left\| \frac{\partial^j U_j}{\partial x^j}(x+\vartheta(y-x),y)(A(y)+c)^{-j} \right\|_{\mathcal{L}(H)} \left\| [t(A(y)+c)]^j e^{-ty^{-2}A(y)} \right\|_{\mathrm{tr}} \\
&\overset{4.3.3,\ (4.12)}{\leqslant} C \left(\sum_{k=0}^{j-a-1} |x-y|^{a+k} t^{\frac{j}{3} - \frac{a+k}{3} - p_0} + |x-y|^j t^{-p_0} \right).
\end{aligned}$$

Multiplying by $e^{-\frac{(x-y)^2}{4t}}$, implies

$$\begin{aligned}
&\left\| e^{-\frac{(x-y)^2}{4t}} t^j (x-y)^a \frac{\partial^a U_j}{\partial x^a}(x,y) e^{-ty^{-2}A(y)} \right\|_{\mathrm{tr}} \\
&\qquad \leqslant C \left(\sum_{k=0}^{j-a-1} e^{-\frac{(x-y)^2}{4t}} |x-y|^{a+k} t^{\frac{j}{3} - \frac{a+k}{3} - p_0} + e^{-\frac{(x-y)^2}{4t}} |x-y|^j t^{-p_0} \right) \\
&\qquad \leqslant C \left(\sum_{k=0}^{j-a-1} t^{\frac{j}{3} + \frac{a+k}{6} - p_0} + t^{\frac{j}{2} - p_0} \right) \overset{a \leqslant j}{\leqslant} C t^{\frac{j}{3} + \frac{a}{6} - p_0}.
\end{aligned} \tag{4.13}$$

Now we are ready to estimate the norm of

$$\int_0^\infty R_s^N(z,y)dz.$$

First, we decompose the integral into three terms as the computations at the beginning of the proof suggest:

$$\begin{aligned}
\int_0^\infty R_s^N(z,y)dz &= -\int_0^\infty \psi''(z) H_s^N(z,y)\varphi(y)dz - 2\int_0^\infty \psi'(z) \frac{\partial H_s^N}{\partial z}(z,y)\varphi(y)dz \\
- \int_0^\infty \psi(z)(4\pi s)^{-\frac{1}{2}} & e^{-\frac{(z-y)^2}{4s}} s^N \left[(z-y)\frac{\partial U_{N+1}}{\partial z}(z,y) + (N+1)U_{N+1}(z,y) \right] e^{-sy^{-2}A(y)}\varphi(y)dz.
\end{aligned}$$

In the first two terms the derivatives $\psi'(z)$ and $\psi''(z)$ are multiplied by $\varphi(y)$. Since ψ has been chosen to be equal to one in a neighbourhood of $\operatorname{supp}\varphi$, this implies

$$\delta := \inf \left\{ |z-y| \mid z \in \operatorname{supp}\psi',\ y \in \operatorname{supp}\varphi \right\} > 0.$$

The first term:

$$\begin{aligned}
&\int_0^\infty \psi''(z)\left\|H_s^N(z,y)\right\|_{\mathrm{tr}}\varphi(y)dz \\
&\qquad \leqslant (4\pi s)^{-\frac{1}{2}}\int_0^\infty \psi''(z)e^{-\frac{(z-y)^2}{4s}}\varphi(y)dz\sum_{j=0}^N s^j \sup_{u\in\operatorname{supp}\psi',\ v\in\operatorname{supp}\varphi}\left\|U_j(u,v)e^{-sv^{-2}A(v)}\right\|_{\mathrm{tr}} \\
&\qquad \overset{(4.12)}{\leqslant} Cs^{-\frac{1}{2}-p_0}e^{-\frac{\delta^2}{4s}}\varphi(y)\int_{\operatorname{supp}\psi''}dz \leqslant Cs^k\varphi(y), \qquad \forall k\in\mathbb{N}.
\end{aligned}$$

The second term:

$$\begin{aligned}
&\int_0^\infty \psi'(z)\left\|\frac{\partial H_s^N}{\partial z}(z,y)\right\|_{\mathrm{tr}}\varphi(y)dz \\
&\qquad \leqslant (4\pi s)^{-\frac{1}{2}}\int_0^\infty \psi'(z)e^{-\frac{(z-y)^2}{4s}}\varphi(y)dz \\
&\qquad\qquad \cdot\sum_{j=0}^N s^j \sup_{u\in\operatorname{supp}\psi',\ v\in\operatorname{supp}\varphi}\left\|\left[-\frac{u-v}{2s}U_j(u,v)+\frac{\partial U_j}{\partial u}(u,v)\right]e^{-sv^{-2}A(v)}\right\|_{\mathrm{tr}} \\
&\qquad \overset{(4.12)}{\leqslant} s^{-\frac{3}{2}-p_0}e^{-\frac{\delta^2}{4s}}\varphi(y)\int_{\operatorname{supp}\psi'}dz \leqslant Cs^k\varphi(y), \qquad \forall k\in\mathbb{N}.
\end{aligned}$$

Both terms do not play any role in the asymptotic expansion. On the other hand, the third term gives

$$\begin{aligned}
&\int_0^\infty \left\|\psi(z)(4\pi s)^{-\frac{1}{2}}e^{-\frac{(z-y)^2}{4s}}s^N\left[(z-y)\frac{\partial U_{N+1}}{\partial z}(z,y)+(N+1)U_{N+1}(z,y)\right]e^{-sy^{-2}A(y)}\varphi(y)\right\|_{\mathrm{tr}}dz \\
&\qquad \leqslant \int_0^\infty (4\pi s)^{-\frac{1}{2}}s^{-1}\Bigg[\left\|\psi(z)e^{-\frac{(z-y)^2}{4s}}s^{N+1}(z-y)\frac{\partial U_{N+1}}{\partial z}(z,y)e^{-sy^{-2}A(y)}\varphi(y)\right\|_{\mathrm{tr}} \\
&\qquad\quad +(N+1)\left\|\psi(z)e^{-\frac{(z-y)^2}{4s}}s^{N+1}U_{N+1}(z,y)e^{-sy^{-2}A(y)}\varphi(y)\right\|_{\mathrm{tr}}\Bigg]dz \\
&\qquad \overset{(4.13)}{\leqslant} C\left(s^{\frac{N+1}{3}+\frac{1}{6}-1-p_0}+s^{\frac{N+1}{3}-1-p_0}\right)\varphi(y)s^{-\frac{1}{2}}\int_0^\infty \psi(z)dz \leqslant Cs^{\frac{N}{3}-\frac{7}{6}-p_0}\varphi(y). \qquad (4.14)
\end{aligned}$$

Taking all estimates together, yields for $N > 3p_0+\frac{1}{2}$

$$\begin{aligned}
\left\|\psi(x)H_t^N(x,y)\varphi(y)-\chi(x)e^{-tT}(x,y)\psi(y)\right\|_{\mathrm{tr}} &\overset{(4.11)}{\leqslant} C\chi(x)\int_0^t\int_0^\infty \left\|R_s^N(z,y)\right\|_{\mathrm{tr}}dzds \\
&\overset{(4.14)}{\leqslant} C\chi(x)\int_0^t s^{\frac{N}{3}-\frac{7}{6}-p_0}ds\varphi(y) \leqslant Ct^{\frac{N}{3}-\frac{1}{6}-p_0}\chi(x)\varphi(y).
\end{aligned}$$

By definition the cut-off functions φ,ψ and χ are equivalent to one on K and the constant C depends on N,ψ and φ. Thus, the assertion is proved. □

4.4 The heat kernel expansion of $T = \partial_x + x^{-1}\varphi(x)S_0$

In this section Theorem 4.3.5 is applied to compute the asymptotic expansion of the heat kernel corresponding to an operator

$$T = \partial_x + x^{-1}\varphi(x)S_0.$$

This section reviews Chapter 4 in [Brü88] and peaks in Theorem 4.4.4 (a more detailed version of Lemma 4.1 in [Brü88]) which is applied in the proof of Theorem 3.4.3.

Let S_0 be the self-adjoint closure of a symmetric elliptic differential operator of first order on a vector bundle E over a closed manifold N. Let $x_1 > 0$ and $\varphi \in C^\infty([0, x_1], (0, 1])$ (Sometimes it will be assumed that there is an $x_0 \in (0, x_1)$, such that $\varphi|_{[0,x_0]} \equiv 1$). Look at the operator

$$T = \frac{\partial}{\partial x} + \frac{\varphi(x)}{x}S_0.$$

Then

$$\begin{aligned}
T^*T &= \left(-\frac{\partial}{\partial x} + \frac{\varphi(x)}{x}S_0\right)\left(\frac{\partial}{\partial x} + \frac{\varphi(x)}{x}S_0\right)\\
&= -\frac{\partial^2}{\partial x^2} - \frac{\varphi'(x)x - \varphi(x)}{x^2}S_0 - \frac{\varphi(x)}{x}S_0\frac{\partial}{\partial x} + \frac{\varphi(x)}{x}S_0\frac{\partial}{\partial x} + \frac{\varphi(x)^2}{x^2}S_0^2\\
&= -\frac{\partial^2}{\partial x^2} + \frac{1}{x^2}\left(\varphi(x)^2S_0^2 + (\varphi(x) - \varphi'(x)x)S_0\right) = -\frac{\partial^2}{\partial x^2} + \frac{1}{x^2}\left(S(x)^2 + \psi(x)S(x)\right)\\
TT^* &= -\frac{\partial^2}{\partial x^2} + \frac{1}{x^2}\left(\varphi(x)^2S_0^2 - (\varphi(x) - \varphi'(x)x)S_0\right) = -\frac{\partial^2}{\partial x^2} + \frac{1}{x^2}\left(S(x)^2 - \psi(x)S(x)\right)
\end{aligned}$$

with

$$S(x) := \varphi(x)S_0 \quad \text{and} \quad \psi(x) := 1 - x\frac{\varphi'(x)}{\varphi(x)}.$$

Definition 4.4.1. For $\varepsilon \in [-1, 1]$ we define

$$A_\varepsilon(x) := S(x)^2 - \varepsilon\psi(x)S(x) : H_1 \to H.$$

Let U_ε be the Friedrichs extension of $-\frac{\partial^2}{\partial x^2} + x^{-2}A_\varepsilon(x)$ with domain $C_0^\infty(\mathbb{R}_+, H_2)$ in $L^2(\mathbb{R}_+, H)$.

A_ε satisfies the assumptions of Definition 4.3.1:

1.

$$\begin{aligned}
\langle A_\varepsilon(x)u, u\rangle &= \left\langle \left[S(x)^2 - \varepsilon\psi(x)S(x)\right]u, u\right\rangle = \left\langle \left[S(x) - \varepsilon\frac{\psi(x)}{2}\mathbb{1}\right]^2 u, u\right\rangle - \left\langle \varepsilon^2\frac{\psi^2(x)}{4}u, u\right\rangle\\
&= \left\| \left[S(x) - \varepsilon\frac{\psi(x)}{2}\mathbb{1}\right]u\right\|^2 - \varepsilon^2\frac{\psi^2(x)}{4}\|u\|^2 \geqslant -\frac{\|\psi\|_\infty^2}{4}\|u\|^2 := (-c+1)\|u\|^2
\end{aligned}$$

2. Let $p_0 > \frac{\dim N}{2}$.

$$\begin{aligned}\sum_{\lambda \in \operatorname{spec} A_\varepsilon(x)} (\lambda + c)^{-p_0} &= \sum_{\lambda \in \operatorname{spec} S_0} \left(\varphi^2(x)\lambda^2 - \varepsilon\psi(x)\varphi(x)\lambda + c\right)^{-p_0} \\ &\leqslant C \sum_{n \in \mathbb{N}} n^{-\frac{2p_0}{\dim N}} < \infty\end{aligned}$$

due to Lemma 1.12.6 on page 113 in [Gil95].

3. Let $k \in \mathbb{N} \cup \{0\}$.

$$\begin{aligned}A_\varepsilon^{(k)}(x) &= \frac{\partial^k}{\partial x^k}\left(\varphi^2(x)S_0^2 + \varepsilon\left(\varphi(x) - x\varphi'(x)\right)S_0\right) \\ &= \frac{\partial^k \varphi^2}{\partial x^k}(x)S_0^2 + \varepsilon\left(\frac{\partial^k \varphi}{\partial x^k}(x) - \frac{\partial^k(x\varphi'(x))}{\partial x_k}\right)S_0\end{aligned}$$

Thus, $A_\varepsilon^{(k)}(x) : H_1 \to H$. Since $\operatorname{supp}\varphi$ is compact, this implies

$$\left\| A_\varepsilon^{(k)}(x)(A_\varepsilon(x) + c)^{-1} \right\|_{\mathcal{L}(H)} \leqslant C_k, \qquad \text{for all } x \in (0, \infty).$$

4. The commutativity of $A_\varepsilon(x)$ and $A_\varepsilon(y)$ follows from the definition since S_0 commutes with itself.

Thus, the assumptions in Theorem 4.3.5 are satisfied, i.e. for every $N \in \mathbb{N}$ exists a constant C_N, such that the trace norm of the diagonal of the heat kernel can be estimated:

$$\left\| e^{-tU_\varepsilon}(x,x) - (4\pi t)^{-\frac{1}{2}} \sum_{j=0}^{N} t^j U_j(x,x) e^{-tx^{-2}A_\varepsilon(x)} \right\|_{\mathrm{tr}} \leqslant C_N t^{\nu_N}. \tag{4.15}$$

Furthermore, $U_j(x,x)$ is a polynomial in $A_\varepsilon(x), A'_\varepsilon(x), \ldots$ of degree $d_j \leqslant \frac{2}{3}j$ with coefficients in $\mathbb{R}[x^{-1}]$. Therefore, it can be written as

$$U_j(x,x) = x^{-2j} \sum_{\substack{k,l \geqslant 0 \\ k+l \leqslant \frac{2}{3}j}} c_{kl}^j(x)\varepsilon^l S^{2k+l}(x) \quad \text{with} \quad c_{kl}^j \in C^\infty((0,\infty), \mathbb{R}). \tag{4.16}$$

If $\varphi|_{(0,x_0]} \equiv 1$, the following lemma will imply that $c_{kl}^j|_{(0,x_0]}$ is constant.

Lemma 4.4.2. *If $\varphi|_{(0,x_0]} \equiv 1$, there are polynomials p_{jk} for $j, k \in \mathbb{N} \cup \{0\}$ of degree $d_{jk} \leqslant \min\left\{\left\lfloor \frac{2}{3}j + \frac{1}{3}k \right\rfloor, j\right\}$, such that*

$$U_{jk}(y) = y^{-2j-k} p_{jk}(S_0^2 - \varepsilon S_0) \qquad \textit{for all } y \in (0, x_0].$$

In particular,

$$U_j(y,y) = U_{j0}(y) = y^{-2j} p_{j0}(S_0^2 - \varepsilon S_0) \qquad \textit{for all } y \in (0, x_0].$$

Proof. We prove the assertion by induction over j. From Lemma 4.3.3 it is known that for all $y \in (0, x_0]$:

$$U_{00}(y) = \mathbb{1} \text{ and } U_{0k}(y) = 0 \text{ for all } k \in \mathbb{N}.$$

Taking $p_{00} \equiv 1$ and $p_{0k} \equiv 0$, proves the case $j = 0$. Let us assume the assertion is proved for a $j \in \mathbb{N} \cup \{0\}$. Looking at the formulas in Lemma 4.3.3 and using the fact that in our case

$$A_\varepsilon(y) = S(y)^2 - \varepsilon\psi(y)S(y) = \varphi^2(y)S_0^2 - \varepsilon\left(1 - y\frac{\varphi'(y)}{\varphi(y)}\right)\varphi(y)S_0 = S_0^2 - \varepsilon S_0.$$

That implies

$$\begin{aligned}
U_{j+1,k}(y) = &\frac{1}{j+k+1}\Bigg[(k+1)(k+2)U_{j,k+2}(y) \\
&\qquad - \sum_{l=0}^{k-1} U_{jl}(y)\sum_{m=0}^{k-l}(-1)^m\frac{m+1}{(k-l-m)!}y^{-m-2}A_\varepsilon^{(k-l-m)}(y)\Bigg] \\
= &\frac{1}{j+k+1}\Bigg[(k+1)(k+2)y^{-2j-k-2}p_{j,k+2}(S_0^2 - \varepsilon S_0) \\
&\qquad - \sum_{l=0}^{k-1} y^{-2j-l}p_{jl}(S_0^2 - \varepsilon S_0)(-1)^{k-l}(k-l+1)y^{-k+l-2}\left(S_0^2 - \varepsilon S_0\right)\Bigg] \\
= &\frac{y^{-2(j+1)-k}}{j+k+1}\Bigg[(k+1)(k+2)p_{j,k+2}(S_0^2 - \varepsilon S_0) \\
&\qquad - \sum_{l=0}^{k-1}(-1)^{k-l}(k-l+1)p_{jl}(S_0^2 - \varepsilon S_0)\left(S_0^2 - \varepsilon S_0\right)\Bigg] \\
=: &\, y^{-2(j+1)-k}p_{j+1,k}(S_0^2 - \varepsilon S_0).
\end{aligned}$$

The degree assertion follows directly from Lemma 4.3.3. $\square$

Inserting Formula (4.16) into Formula (4.15), shows that a typical term in the expansion is

$$S(x)^{2k+l}e^{-tx^{-2}A_\varepsilon(x)} = S(x)^{2k+l}e^{-tx^{-2}\left(S(x)^2 - \varepsilon\psi(x)S(x)\right)}.$$

Its norm is estimated with the help of Taylor's theorem

$$\begin{aligned}
&\left\| S(x)^{2k+l}e^{-tx^{-2}\left(S(x)^2 - \varepsilon\psi(x)S(x)\right)} - \sum_{m=0}^{M-1}\frac{(t\varepsilon x^{-2}\psi(x))^m}{m!}S(x)^{2k+l+m}e^{-tx^{-2}S(x)^2}\right\|_{\mathrm{tr}} \\
&\overset{\exists\ \vartheta\in[0,1]}{=} \frac{(t\varepsilon x^{-2}\psi(x))^M}{M!}\left\| S(x)^{2k+l+M}e^{-tx^{-2}\left(S(x)^2 - \vartheta\varepsilon\psi(x)S(x)\right)}\right\|_{\mathrm{tr}} \\
&\leqslant C\frac{(\varepsilon x^{-2}\psi(x))^M}{M!}t^{\frac{M}{2} - \frac{2k+l}{2} - \frac{\dim N + 1}{2}}
\end{aligned}$$

and thus yields

$$\Bigg\| e^{-tU_\varepsilon}(x,x) - (4\pi)^{-\frac{1}{2}} \sum_{j=0}^{N} \sum_{\substack{k,l\geqslant 0\\ k+l\leqslant \frac{2}{3}j}} \sum_{m=0}^{M_N} t^{j+m-\frac{1}{2}} \varepsilon^{l+m} c^j_{kl}(x) \frac{x^{-2m-2j}\psi(x)^m}{m!} S(x)^{2k+l+m} e^{-tx^{-2}S(x)^2} \Bigg\|_{\mathrm{tr}} \leqslant Ct^{\nu_N}$$

with $\nu_N \to \infty$. Inserting $\varepsilon = \pm 1$ and taking the difference, leads to cancellations

$$\Bigg| \operatorname{tr}_H e^{-tT^*T}(x,x) - \operatorname{tr}_H e^{-tTT^*}(x,x) - \pi^{-\frac{1}{2}} \sum_{j=0}^{N} \sum_{\substack{k,l\geqslant 0\\ k+l\leqslant \frac{2}{3}j}} \sum_{\substack{m=0\\ l+m \text{ odd}}}^{M_N} t^{j+m-\frac{1}{2}} c^j_{kl}(x) \frac{x^{-2m-2j}\psi(x)^m}{m!} \operatorname{tr}_H \left(S(x)^{2k+l+m} e^{-tx^{-2}S(x)^2} \right) \Bigg| \leqslant Ct^{\nu_N}. \tag{4.17}$$

We need to compute

$$\operatorname{tr}_H \left(S(x)^{2k+l+m} e^{-tx^{-2}S(x)^2} \right),$$

where $2k+l+m$ is odd since $l+m$ is odd.

Lemma 4.4.3. *For $q \in \mathbb{N} \cup \{0\}$ we have*

$$\operatorname{tr}_H \left(S(x)^{2q+1} e^{-tx^{-2}S(x)^2} \right) \sim_{t\to 0} \sum_{n=1}^{\dim N+2q+1} \frac{1}{2} x^n \varphi(x)^{-n+2q+1} \Gamma\left(\frac{n}{2}\right) \operatorname{Res}_{n-2q-1} \eta_{S_0} t^{-\frac{n}{2}} + \sum_{n=0}^{\infty} \frac{1}{2} x^{-n} \varphi(x)^{n+2q+1} b^q_{n+\dim N+2q+1} t^{\frac{n}{2}}.$$

The constants b^q_n depend on values and residua of the eta function. They are explicitly given in the proof.

Proof. Let $P := x^{-1}S(x) = x^{-1}\varphi(x)S_0$ and $m := \dim N$.

$$\begin{aligned}
\eta_P(s) &= \sum_{\lambda\in\operatorname{spec} P} \operatorname{sgn}\lambda |\lambda|^{-s} = \Gamma\left(\frac{s+2q+1}{2}\right)^{-1} \sum_{\lambda\in\operatorname{spec} P} \int_0^\infty \left(\frac{t}{\lambda^2}\right)^{\frac{s+2q+1}{2}-1} \lambda^{2q+1} e^{-t} \frac{dt}{\lambda^2} \\
&= \Gamma\left(\frac{s+2q+1}{2}\right)^{-1} \int_0^\infty t^{\frac{s+2q+1}{2}-1} \sum_{\lambda\in\operatorname{spec} P} \lambda^{2q+1} e^{-t\lambda^2} dt \\
&= \Gamma\left(\frac{s+2q+1}{2}\right)^{-1} \int_0^\infty t^{\frac{s+2q+1}{2}-1} \operatorname{tr}_H(P^{2q+1}e^{-tP^2}) dt \\
&= \zeta\left(\frac{s+2q+1}{2}, P^2, P^{2q+1}\right) \qquad \text{by Formula (1.12.47) on page 112 in [Gil95].}
\end{aligned}$$

Lemma 1.12.5 on page 112, Formula (1.13.1) and Formula (1.13.2) on page 114 of [Gil95] give

$$\begin{aligned}
&a_n(P^2, P^{2q+1}) = \operatorname{Res}_{s=\frac{m-n+2q+1}{2}} \left(\Gamma(s)\zeta(s, P^2, P^{2q+1})\right) \\
&= \frac{1}{2}\operatorname{Res}_{s=m-n}\left(\Gamma\left(\frac{s+2q+1}{2}\right)\zeta\left(\frac{s+2q+1}{2}, P^2, P^{2q+1}\right)\right) \\
&= \frac{1}{2}\operatorname{Res}_{s=m-n}\left(\Gamma\left(\frac{s+2q+1}{2}\right)\eta_P(s)\right) \\
&= \frac{1}{2}\left(\frac{\varphi(x)}{x}\right)^{n-m}\operatorname{Res}_{s=m-n}\left(\Gamma\left(\frac{s+2q+1}{2}\right)\eta_{S_0}(s)\right) \\
&= \frac{1}{2}\left(\frac{\varphi(x)}{x}\right)^{n-m}\begin{cases}\Gamma\left(\frac{m-n+2q+1}{2}\right)\operatorname{Res}_{m-n}\eta_{S_0} & n = 0,1,\dots,m+2q \\ \operatorname{Res}_{s=-2l-1}\Gamma\left(\frac{s+1}{2}\right)\eta_{S_0}(-2q-2l-1) & \\ \quad = 2\frac{(-1)^l}{l!}\eta_{S_0}(-2q-2l-1) & n = m+2q+2l+1, l \in \mathbb{N}\cup\{0\} \\ \Gamma\left(-l+\frac{1}{2}\right)\operatorname{Res}_{-2q-2l}\eta_{S_0} & n = m+2q+2l, l\in\mathbb{N}\end{cases} \\
&=: \frac{1}{2}\left(\frac{\varphi(x)}{x}\right)^{n-m} b_n^q, \qquad n \in \mathbb{N}\cup\{0\}\,.
\end{aligned}$$

The second to last equality follows from the fact that the eta function may have poles at $s = m, m-1, \dots 3, 2, 1, -2, -4, -6, -8, \dots$. This follows from:

- $\Gamma(\frac{s+1}{2})\eta_{S_0}(s)$ has at most simple poles at $s = m, m-1, \dots$. (This follows directly from Theorem 1.12.5 and Formula (1.13.2) in [Gil95]. Disregard Lemma 1.13.1 b), this is a misprint.)
- $\Gamma(\frac{s+1}{2})$ has poles at $-1, -3, \dots$ and is nowhere zero.
- By Theorem 3.8.1 on page 284 in [Gil95] the eta function is regular at 0.

From Lemma 1.9.1 on page 75 in [Gil95] follows

$$\begin{aligned}
\operatorname{tr}_H\left(S(x)^{2q+1}e^{-tx^{-2}S(x)^2}\right) &= x^{2q+1}\operatorname{tr}_H\left[P^{2q+1}e^{-tP^2}\right] \\
&\sim_{t\to 0} x^{2q+1}\sum_{n=0}^{\infty} a_n(P^2, P^{2q+1})t^{\frac{n-m-2q-1}{2}} \\
&\sim_{t\to 0} x^{2q+1}\sum_{n=0}^{m+2q}\frac{1}{2}\left(\frac{\varphi(x)}{x}\right)^{n-m}\Gamma\left(\frac{m-n+2q+1}{2}\right)\operatorname{Res}_{m-n}\eta_{S_0}t^{\frac{n-m-2q-1}{2}} \\
&\qquad + x^{2q+1}\sum_{n=m+2q+1}^{\infty}\frac{1}{2}\left(\frac{\varphi(x)}{x}\right)^{n-m} b_n^q t^{\frac{n-m-2q-1}{2}} \\
&\sim_{t\to 0} \sum_{n=1}^{m+2q+1}\frac{1}{2}x^n\varphi(x)^{-n+2q+1}\Gamma\left(\frac{n}{2}\right)\operatorname{Res}_{n-2q-1}\eta_{S_0}t^{-\frac{n}{2}} \\
&\qquad + \sum_{n=0}^{\infty}\frac{1}{2}x^{-n}\varphi(x)^{n+2q+1}b^q_{n+m+2q+1}t^{\frac{n}{2}}.
\end{aligned}$$

□

Theorem 4.4.4. [16] *For $x > 0$ we have an asymptotic expansion*

$$\operatorname{tr}_H e^{-tT^*T}(x,x) - \operatorname{tr}_H e^{-tTT^*}(x,x) \sim_{t\to 0} \sum_{p=0}^{\dim N} \left[\sum_{n=p}^{\dim N} g_{pn}(x) \operatorname{Res}_n \eta_{S_0} \right] x^{p-1} t^{-\frac{p}{2}} + O_x(t^{\frac{1}{2}}).$$

The coefficients $g_{pn} \in C^\infty(\mathbb{R}_+, \mathbb{R})$ are given by

$$g_{pn}(x) = (4\pi)^{-\frac{1}{2}} \varphi(x)^{-n} \underbrace{\sum_{\substack{j,k,l\geqslant 0 \\ k+l\leqslant \frac{2}{3}j}} \sum_{\substack{m\geqslant 0 \\ l+m \text{ odd}}}}_{2k+l-2j-m+1=p-n} c^j_{kl}(x) \frac{\psi(x)^m}{m!} \Gamma\left(\frac{n+2k+l+m}{2}\right).$$

If $\varphi|_{(0,x_0]} \equiv 1$, then $g_{pn}|_{(0,x_0]} \equiv g_{pn}(x_0)$.

Proof. Inserting the formulas of Lemma 4.4.3 into (4.17), gives the asymptotic expansion:

$$\begin{aligned}
&\operatorname{tr}_H e^{-tT^*T}(x,x) - \operatorname{tr}_H e^{-tTT^*}(x,x) \\
&\sim_{t\to 0} \pi^{-\frac{1}{2}} \sum_{\substack{j,k,l\geqslant 0 \\ k+l\leqslant \frac{2}{3}j}} \sum_{\substack{m\geqslant 0 \\ l+m \text{ odd}}} t^{j+m-\frac{1}{2}} c^j_{kl}(x) \frac{x^{-2m-2j}\psi(x)^m}{m!} \\
&\qquad \cdot \left[\sum_{n=1}^{\dim N+2k+l+m} \frac{1}{2} x^n \varphi(x)^{-n+2k+l+m} \Gamma\left(\frac{n}{2}\right) \operatorname{Res}_{n-2k-l-m} \eta_{S_0} t^{-\frac{n}{2}} \right. \\
&\qquad\qquad \left. + \sum_{n=0}^{\infty} \frac{1}{2} x^{-n} \varphi(x)^{n+2k+l+m} b^{\frac{2k+l+m-1}{2}}_{n+\dim N+2k+l+m} t^{\frac{n}{2}} \right] \\
&\overset{j+m\geqslant 1}{\sim_{t\to 0}} \pi^{-\frac{1}{2}} \sum_{\substack{j,k,l\geqslant 0 \\ k+l\leqslant \frac{2}{3}j}} \sum_{\substack{m\geqslant 0 \\ l+m \text{ odd}}} \sum_{n=1}^{\dim N+2k+l+m} c^j_{kl}(x) \frac{x^{-2m-2j+n}\psi(x)^m}{m!} \\
&\qquad \cdot \frac{1}{2} \varphi(x)^{-n+2k+l+m} \Gamma\left(\frac{n}{2}\right) \operatorname{Res}_{n-2k-l-m} \eta_{S_0} t^{-\frac{n+1}{2}+j+m} + O_x(t^{\frac{1}{2}}) \\
&\sim_{t\to 0} (4\pi)^{-\frac{1}{2}} \sum_{\substack{j,k,l\geqslant 0 \\ k+l\leqslant \frac{2}{3}j}} \sum_{\substack{m\geqslant 0 \\ l+m \text{ odd}}} \sum_{n=1-2k-l-m}^{\dim N} c^j_{kl}(x) \frac{\psi(x)^m}{m!} \Gamma\left(\frac{n+2k+l+m}{2}\right) \\
&\qquad \cdot \varphi(x)^{-n} \operatorname{Res}_n \eta_{S_0} x^{n+2k+l-2j-m} t^{-\frac{n+2k+l-2j-m+1}{2}} + O_x(t^{\frac{1}{2}}) \\
&\sim_{t\to 0} \sum_{p=0}^{\dim N} \sum_{n=p}^{\dim N} \left[\underbrace{\sum_{\substack{j,k,l\geqslant 0 \\ k+l\leqslant \frac{2}{3}j}} \sum_{\substack{m\geqslant 0 \\ l+m \text{ odd}}}}_{2k+l-2j-m+1=p-n} (4\pi)^{-\frac{1}{2}} c^j_{kl}(x) \frac{\psi(x)^m}{m!} \Gamma\left(\frac{n+2k+l+m}{2}\right) \right. \\
&\qquad \left. \cdot \varphi(x)^{-n} \right] \operatorname{Res}_n \eta_{S_0} x^{p-1} t^{-\frac{p}{2}} + O_x(t^{\frac{1}{2}})
\end{aligned}$$

[16] This is an explicit version of Theorem 4.1 in [Brü88]. The explicit calculations were possible due to the direct computation of the trace demonstrated in the proof of Lemma 4.4.3.

$$\sim_{t\to 0} \sum_{p=0}^{\dim N} \left[\sum_{n=p}^{\dim N} g_{pn}(x) \operatorname{Res}_n \eta_{S_0} \right] x^{p-1} t^{-\frac{p}{2}} + O_x(t^{\frac{1}{2}}),$$

where we have used that

$$n + 2k + l - 2j - m + 1 \leqslant \dim N + 2((k+l) - j) - (m+l) + 1 \leqslant \dim N$$

since $m + l$ is odd and $k + l \leqslant \frac{2}{3}j$. For $n = \dim N,\ j = k = l = 0,\ m = 1$ the inequality becomes an equality. Furthermore, the double sum has only finitely many terms:

$$0 \leqslant p = n + 2k + l - 2j - m + 1 \leqslant \dim N + 2 \cdot \frac{2}{3}j - 2j - m + 1 = \dim N - \frac{2}{3}j - m + 1$$
$$\Rightarrow \quad \frac{2}{3}j + m \leqslant \dim N + 1 \qquad \Rightarrow \quad j \leqslant \left\lfloor \frac{3}{2}(\dim N + 1) \right\rfloor, \quad m \leqslant \dim N + 1.$$

Next, $g_{pn}(x) \in C^\infty(\mathbb{R}_+, \mathbb{R})$ due to the properties of $\varphi,\ \psi$ and c^j_{kl}.

If $\varphi|_{(0,x_0]} \equiv 1$, the assertion $g_{pn}|_{(0,x_0]} \equiv g_{pn}(x_0)$ follows from

- $\psi(x) = 1 - x\frac{\varphi'(x)}{\varphi(x)} = 1$ for all $x \in (0, x_0]$ and
- $c^j_{kl}(x) = c^j_{kl}(x_0)$ for all $x \in (0, x_0]$ due to Lemma 4.4.2.

□

5 The index of Spin-Dirac, Gauss-Bonnet and Signature operator on manifolds with multiply warped product singularities

5.1 Manifolds with multiply warped product singularities

Definition 5.1.1. For $l \in \mathbb{N}$, let $(N_1, g_1), \ldots, (N_l, g_l)$ be Riemannian manifolds with $n_i := \dim N_i,\ i = 1, \ldots, l$ and $h_1, \ldots, h_l \in C^1((0, s_0], (0, \infty))$ with $h_1(s_0) = \ldots = h_l(s_0) = 1$. The Riemannian manifold

$$U = (0, s_0] \times \underbrace{N_1 \times \cdots \times N_l}_{=:N} \quad \text{with metric} \quad g(r) = dr^2 \oplus h_1^2(r) g_1 \oplus \cdots \oplus h_l^2(r) g_l$$

is called a *multiply warped product.*

A *manifold with a multiply warped product singularity* is a Riemannian manifold (M, g) that can be decomposed into a compact manifold M_1 with boundary N and a multiply warped product $U = (0, s_0] \times N$, such that the boundary of M_1 is equal to $\{s_0\} \times N$.

We introduce the following notations:

- Let ∇ denote the Levi-Civita connection on M. Furthermore, let E be a bundle over M with connection ∇^E.
- The index sets:
$$I_q = \left\{ \sum_{i=1}^{q-1} n_i + 1, \ldots, \sum_{i=1}^{q} n_i \right\} \text{ for all } 1 \leqslant q \leqslant l.$$
- Products of warping functions: For $I \subset \{1, \ldots, n\}$ define $h_I := \prod_{i \in I} h_i$ and $h := \prod_{i=1}^{n} h_i$. For $i \in \{1, \ldots, n\}$ let $h_{q(i)} := h_p$ if $i \in I_p$.

Examples: If $l = 1$, then $U = \big((0, s_0] \times N, dr^2 \oplus h^2(r) g\big)$ is known as *warped product.* Two special cases are:

1. $h(r) = r$ close to $r = 0$ describes a *conic singularity* and
2. $h(r) = r^\beta,\ \beta > 1$ close to $r = 0$ describes a *metric horn singularity.*

Let $\gamma_x(t) := (t, x) \in U$, $X \in T_{(s_0, x)}U$ and $\sigma \in E_{s_0, x}$:

$$\overline{X}(r, x) := \left(\mathscr{P}^{\nabla}_{\gamma_x} X\right)(r, x), \quad \overline{\sigma}(r, x) := \left(\mathscr{P}^{\nabla^E}_{\gamma_x} \sigma\right)(r, x), \quad \partial_r(r, x) = \left(\mathscr{P}^{\nabla}_{\gamma_x}(1, 0)\right)(r, x).$$

Lemma 5.1.2. *Let $B^N = (e_i)_{i=1}^n$ be the union of local orthonormal frames $(e_i)_{i\in I_q}$ on a neighbourhood U_q in N_q for all $q = 1,\dots,l$. Then $B := \{\partial_r\} \cup \{\overline{e_i}\}_{i=1}^n$ is a local orthonormal frame on $(0, s_0) \times U_1 \times \dots \times U_l$ in (U, g).*

Proof. $\overline{e_i} = \mathscr{P}^{\nabla}_{\partial_r} e_i$, i.e. $\nabla_{\partial_r}\overline{e_i} = 0$. Since ∇ is the Levi-Civita connection,

$$\partial_r g(\overline{e_i}, \overline{e_j}) = g(\nabla_{\partial_r}\overline{e_i}, \overline{e_j}) + g(\overline{e_i}, \nabla_{\partial_r}\overline{e_j}) = 0.$$

$$\Rightarrow \qquad g_{(r,x)}(\overline{e_i}, \overline{e_j}) = g_{(s_0,x)}(e_i, e_j) \equiv \delta_{ij}$$

Next, we prove $g(\partial_r, \overline{e_i})_{(r,x)} = 0$. Let $X \in T_{s_0,x}U$ and decompose its parallel-transported version $\overline{X}$ into normal coordinates:

$$\overline{X} = v^0\partial_r + \sum_{k=1}^n v^k \frac{\partial}{\partial x_k}.$$

The functions $v^k, k = 0,\dots,n$ are uniquely defined by the differential equations of the parallel transport along $\gamma_x(r) = (r, x)$ ([dC92] page 53):

$$0 = \frac{dv^k}{dr} + \sum_{i,j=0}^n \Gamma^k_{ij} v^j \frac{dx_i(\gamma_x(r))}{dr} \overset{x_0=r}{=} \frac{dv^k}{dr} + \sum_{j=0}^n \Gamma^k_{0j} v^j.$$

We are interested in the case $k = 0$:

$$0 = \frac{dv^0}{dr} + \sum_{j=0}^n \Gamma^0_{0j} v^j.$$

We calculate the appropriate Christoffel symbols ([dC92] page 56):

$$\Gamma^0_{0j} = \frac{1}{2}\sum_{m=0}^n \left(\frac{\partial}{\partial r} g_{mj} + \frac{\partial}{\partial x_j} \underbrace{g_{0m}}_{=\delta_{0m}} - \frac{\partial}{\partial x_m} \underbrace{g_{0j}}_{=\delta_{0j}} \right) \underbrace{g^{0m}}_{=\delta_{0m}} = \frac{1}{2}\frac{\partial g_{0j}}{\partial r} = \frac{1}{2}\frac{\partial \delta_{0j}}{\partial r} = 0.$$

Thus, $\frac{d}{dr}v^0 = 0$ and

$$g(X, \partial_r)_{(s_0,x)} = 0 \quad \Rightarrow \quad g(\overline{X}, \partial_r)_{(r,x)} = 0 \; \forall r \in (0, s_0].$$

□

Lemma 5.1.3. *The basis vector fields are given by*

$$E_i(r, x) := \overline{e_i}(r, x) = \frac{1}{h_q(r)} e_i(x), \qquad i \in I_q \text{ and } q = 1,\dots,l.$$

Proof. We choose normal coordinates $(\widetilde{U}, x_1,\dots,x_n)$ on $N = N_1 \times \dots \times N_l$. Then $((0, s_0 + \varepsilon) \times \widetilde{U}, (x_0, x_1,\dots,x_n))$ with $x_0 = r$ defines a chart on U. For $i \in I_q$ we decompose E_i:

$$E_i = \sum_{k=0}^n v_k \frac{\partial}{\partial x_k}.$$

To find v_k, we compute the Christoffel symbols ([dC92] page 56):

$$\Gamma_{0j}^k := \frac{1}{2}\sum_{m=0}^n \left\{ \frac{\partial}{\partial r} g_{jm} + \frac{\partial}{\partial x_j} \underbrace{g_{m0}}_{=\delta_{m0}} - \frac{\partial}{\partial x_m} \underbrace{g_{0j}}_{\delta_{0j}} \right\} g^{mk} = \frac{1}{2}\sum_{m=0}^n \left\{ \frac{\partial}{\partial r} g_{jm} \right\} g^{mk}$$

$$g_{jm}(r,x) = \begin{cases} h_q(r)^2 g_{q,jm}(x_{I_q}) = h_q(r)^2 \delta_{jm} & j,m \in I_q \text{ for a } q \in \{1,\dots,l\} \\ 1 & j=m=0, \\ 0 & \text{else.} \end{cases}$$

$$\Rightarrow \qquad \Gamma_{0j}^k = \frac{h_q'}{h_q}\delta_{jk} \quad \text{if} \quad k \in I_q.$$

Inserting into

$$0 = \frac{dv_k}{dr} + \sum_{j=0}^n \Gamma_{0j}^k v_j,$$

gives

$$0 = \frac{dv_k}{dr} + \sum_{j=1}^n \frac{h_{q(k)}'}{h_{q(k)}} \delta_{jk} v_j = v_k' + \frac{h_{q(k)}'}{h_{q(k)}} v_k.$$

Thus,

$$\begin{aligned} v_k &= \frac{1}{h_{q(k)}(r)} v_k(s_0,x) = 0 && k \neq i \text{ and} \\ v_i &= \frac{1}{h_{q(k)}(r)} v_i(s_0,x) = \frac{1}{h_{q(k)}(r)}. \end{aligned}$$

□

5.2 The index of the Spin-Dirac operator

In this section, we assume that M is an even-dimensional spin manifold and $S(M)$ an irreducible complex spinor bundle over M (as defined on page 121 in [LM89]). In this case the spin connection ∇^S and the covariant derivative on the tangent bundle are connected by a simple formula. Therefore, we start off by calculating the Levi-Civita connection ∇ on U with the help of Lemma 5.1.3. The computations here are a generalization of the computations in [Cho85].

Lemma 5.2.1. *On (U,g) the Levi-Civita connection forms are given by:*

$$\begin{aligned} \omega_{ij}(E_k) &= \frac{1}{h_q}\omega_{ij}^q(e_k) && i,j,k \in I_q,\ q=1,\dots,l \qquad \textit{and} \\ \omega_{0i}(E_i) = -\omega_{i0}(E_i) &= \frac{\partial \log h_q}{\partial r} && i \in I_q,\ q = 1,\dots,l, \end{aligned}$$

where ω_{ij}^q are the connection forms of the Levi-Civita connection on N_q. All the other connection forms are zero.

Proof. The connection forms are defined by

$$\nabla E_i = \sum_{j=0}^{n} \omega_{ij} \otimes E_j, \ \nabla^q e_i = \sum_{j \in I_q} \omega_{ij}^q \otimes e_j \text{ for } i \in I_q \text{ and } q = 1, \dots, l.$$

They can be calculated by using the *Koszul formula* ([dC92] page 55)

$$\begin{aligned} 2g(\nabla_X Y, Z) =& Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\ &+ g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X). \end{aligned}$$

Let $i \in I_q, j \in I_p$:

$$[E_i, E_j] = \frac{1}{h_q} e_i \frac{1}{h_p} e_j - \frac{1}{h_p} e_j \frac{1}{h_q} e_i = \frac{1}{h_q h_p} [e_i, e_j] = \frac{\delta_{qp}}{h_q^2} [e_i, e_j] \in TN_q.$$

This implies

$$g([E_i, E_j], E_k) = \begin{cases} \dfrac{1}{h_q} g_q([e_i, e_j], e_k) & \text{if } i, j, k \in I_q \text{ for a } q \in \{1, \dots, l\}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the Koszul formula implies that $\omega_{ij}(E_k) = 0$ if i, j, k are not in the same index set I_q. Let $i, j, k \in I_q$ and insert the above result into the Koszul formula:

$$\begin{aligned} 2g(\nabla_{E_k} E_i, E_j) =& g([E_k, E_i], E_j) - g([E_k, E_j], E_i) - g([E_i, E_j], E_k) \\ =& \frac{1}{h_q} g_q([e_k, e_i], e_j) - \frac{1}{h_q} g_q([e_k, e_j], e_i) - \frac{1}{h_q} g_q([e_i, e_j], e_k) \\ =& \frac{2}{h_q} g_q(\nabla^q_{e_k} e_i, e_j). \end{aligned}$$

Since $\nabla_{\partial_r} E_i = 0$, the cases $i, j \in \{1, \dots, n\}$ and $k \in \{0, 1, \dots, n\}$ are clear. Let $j \in I_p$ and $k \in I_q$:

$$\begin{aligned} 2g(\nabla_{E_k} \partial_r, E_j) &= g([E_k, \partial_r], E_j) - g([E_k, E_j], \partial_r) - g([\partial_r, E_j], E_k) \\ &= g\left(-\frac{\partial}{\partial r}\left(\frac{1}{h_q}\right) e_k, E_j\right) - 0 - g\left(\frac{\partial}{\partial r}\left(\frac{1}{h_p}\right) e_j, E_k\right) \\ &= \frac{h_q'}{h_q} g(E_k, E_j) + \frac{h_p'}{h_p} g(E_j, E_k) = 2\frac{\partial \log h_q}{\partial r} \delta_{jk}. \end{aligned}$$

Since $g(\nabla_{\partial_r} \partial_r, E_j) = \partial_r g(\partial_r, E_j) - g(\partial_r, \nabla_{\partial_r} E_j) = 0$, we have calculated all connection forms. $\square$

Lemma 5.2.2. *Let $(\sigma_\alpha)_\alpha$ be a local spinor basis of $S(N)$ determined by $e_1, \dots, e_n$.*

$$\nabla^S_{E_i} \overline{\sigma_\alpha} = \frac{1}{h_q} \overline{\tilde{\nabla}^q_{e_i} \sigma_\alpha} + \frac{1}{2} \frac{\partial \log h_q}{\partial r} c(\partial_r) c(E_i) \cdot \overline{\sigma_\alpha} \quad \textit{for } i \in I_q \textit{ and } q = 1, \dots, l,$$

where $\widetilde{\nabla}^q$ *is given by*

$$\widetilde{\nabla}^q_{e_k}\sigma_\alpha = \frac{1}{2}\sum_{\substack{i,j\in I_q\\ i<j}} \omega^q_{ij}(e_k)c(e_i)c(e_j)\sigma_\alpha.$$

Proof. Due to Theorem 4.15 on page 110 in [LM89], we find for $i \in I_q$:

$$\begin{aligned}
\nabla^S_{E_i}\overline{\sigma_\alpha} =& \frac{1}{2}\sum_{k=1}^n \omega_{0k}(E_i)c(\partial_r)c(E_k)\cdot\overline{\sigma_\alpha} + \frac{1}{2}\sum_{\substack{j,k=1\\ j<k}}^n \omega_{jk}(E_i)c(E_j)c(E_k)\cdot\overline{\sigma_\alpha}\\
=& \frac{1}{2}\frac{\partial \log h_q}{\partial r}c(\partial_r)c(E_i)\cdot\overline{\sigma_\alpha} + \frac{1}{2}\sum_{\substack{j,k\in I_q\\ j<k}} \frac{1}{h_q}\omega^q_{jk}(e_i)c(E_j)c(E_k)\cdot\overline{\sigma_\alpha}\\
=& \frac{1}{2}\frac{\partial \log h_q}{\partial r}c(\partial_r)c(E_i)\cdot\overline{\sigma_\alpha} + \frac{1}{h_q}\overline{\frac{1}{2}\sum_{\substack{j,k\in I_q\\ j<k}} \omega^q_{jk}(e_i)c(e_j)c(e_k)\cdot\sigma_\alpha}\\
=& \frac{1}{2}\frac{\partial \log h_q}{\partial r}c(\partial_r)c(E_i)\cdot\overline{\sigma_\alpha} + \frac{1}{h_q}\overline{\widetilde{\nabla}^q_{e_i}\sigma_\alpha}.
\end{aligned}$$

□

Lemma 5.2.3. *Under the unitary map*

$$\begin{aligned}
\Psi : L^2\left(I, \Gamma(S(N)), g_1\oplus\cdots\oplus g_l\right) &\to L^2\left(S^+(U), g\right)\\
\sigma_\alpha(x) &\mapsto h^{-\frac{1}{2}}(r)\frac{1+c(\partial_r)}{2}\overline{\sigma}_\alpha(r,x)
\end{aligned}$$

the Spin-Dirac operator transforms to

$$\Psi^{-1}(-c(\partial_r)D|_U)\Psi = \frac{\partial}{\partial r} - \sum_{q=1}^l \frac{1}{h_q}\widetilde{D}_q,$$

where $\widetilde{D}_q := \sum_{i\in I_q} c(e_i)\widetilde{\nabla}^q_{e_i}$.

Proof. We compute

$$\begin{aligned}
(-c(\partial_r)D|_U)\, f(r)\overline{\sigma_\alpha} &= -c(\partial_r)\left(c(\partial_r)\nabla^S_{\partial_r} + \sum_{i=1}^n c(E_i)\nabla^S_{E_i}\right) f(r)\overline{\sigma_\alpha}\\
&= \left(\nabla^S_{\partial_r} - c(\partial_r)\sum_{q=1}^l\sum_{i\in I_q} c(E_i)\nabla^S_{E_i}\right) f(r)\overline{\sigma_\alpha}\\
&\overset{5.2.2}{=} f'(r)\overline{\sigma_\alpha} - f(r)c(\partial_r)\sum_{q=1}^l\sum_{i\in I_q} c(E_i)\frac{1}{h_q}\overline{\widetilde{\nabla}^q_{e_i}\sigma_\alpha}
\end{aligned}$$

$$
\begin{aligned}
&- f(r)\frac{1}{2}\sum_{q=1}^{l}\sum_{i\in I_q}\frac{\partial \log h_q}{\partial r}c(\partial_r)c(E_i)c(\partial_r)c(E_i)\overline{\sigma_\alpha} \\
&= f'(r)\overline{\sigma_\alpha} - f(r)c(\partial_r)\sum_{q=1}^{l}\frac{1}{h_q}\overline{\sum_{i\in I_q} c(e_i)\widetilde{\nabla}^q_{e_i}\sigma_\alpha} + f(r)\frac{1}{2}\sum_{q=1}^{l} n_q \frac{\partial \log h_q}{\partial r}\overline{\sigma_\alpha} \\
&= f'(r)\overline{\sigma_\alpha} - f(r)c(\partial_r)\sum_{q=1}^{l}\frac{1}{h_q}\overline{\widetilde{D}_q\sigma_\alpha} + f(r)\frac{\partial}{\partial r}\left[\log\sqrt{h_1^{n_1}\cdots h_l^{n_l}}\right]\overline{\sigma_\alpha} \\
&= f'(r)\overline{\sigma_\alpha} - f(r)c(\partial_r)\sum_{q=1}^{l}\frac{1}{h_q}\overline{\widetilde{D}_q\sigma_\alpha} + f(r)\frac{\partial \log\sqrt{h}}{\partial r}\overline{\sigma_\alpha}.
\end{aligned}
$$

$$
\begin{aligned}
&\Psi^{-1}(-c(\partial_r)D|_U)\Psi f(r)\sigma_\alpha = \Psi^{-1}(-c(\partial_r)D|_U)\left(\frac{f(r)}{\sqrt{h(r)}}\frac{1+c(\partial_r)}{2}\overline{\sigma_\alpha}\right) \\
&= \Psi^{-1}\Bigg(\frac{\partial h^{-\frac{1}{2}}(r)}{\partial r}f(r) + \frac{f'(r)}{\sqrt{h(r)}} + \frac{f(r)}{\sqrt{h(r)}}\frac{\partial \log\sqrt{h(r)}}{\partial r} \\
&\qquad\qquad - \frac{f(r)}{\sqrt{h(r)}}c(\partial_r)\sum_{q=1}^{l}\frac{1}{h_q(r)}\overline{\widetilde{D}_q}\Bigg)\frac{1+c(\partial_r)}{2}\overline{\sigma_\alpha} \\
&= \Psi^{-1}\frac{1}{\sqrt{h(r)}}\left(\frac{1+c(\partial_r)}{2}f'(r)\overline{\sigma_\alpha} - f(r)c(\partial_r)\frac{1-c(\partial_r)}{2}\sum_{q=1}^{l}\frac{1}{h_q(r)}\overline{\widetilde{D}_q\sigma_\alpha}\right) \\
&= \Psi^{-1}\frac{1}{\sqrt{h(r)}}\frac{1+c(\partial_r)}{2}\overline{\left(f'(r)\sigma_\alpha - f(r)\sum_{q=1}^{l}\frac{1}{h_q(r)}\widetilde{D}_q\sigma_\alpha\right)} \\
&= \left(f'(r) - f(r)\sum_{q=1}^{l}\frac{1}{h_q(r)}\widetilde{D}_q\right)\sigma_\alpha = \left(\frac{\partial}{\partial r} - \sum_{q=1}^{l}\frac{1}{h_q(r)}\widetilde{D}_q\right)f(r)\sigma_\alpha
\end{aligned}
$$

□

Lemma 5.2.4. *If $N_i,\ i = 1,\dots,l$ are spin manifolds and $N = N_1\times\cdots\times N_l$ is equipped with the product spin structure, the elliptic symmetric first order differential operators*

$$T(r) := \sum_{q=1}^{l}\frac{1}{h_q}\widetilde{D}_q, \qquad \textit{for } r\in(0,s_0]$$

satisfy

$$T(r)^2\sigma_1\otimes\cdots\otimes\sigma_l = \sum_{q=1}^{l}\frac{1}{h_q^2}\sigma_1\otimes\cdots\otimes D_q^2\sigma_q\otimes\cdots\otimes\sigma_l,$$

where D_q is the Spin-Dirac operator on $S(N_q)$.
$T(r)$ is self-adjoint in $\mathcal{L}(H^1(N,S(N)),L^2(N,S(N)))$ and $\ker T(r) = \bigotimes_{q=1}^{l}\ker D_q$ for all $r\in I$.

Proof. If N_i is a spin manifold for every $i = 1, \dots, l$, the product $N = N_1 \times \dots \times N_l$ is canonically a spin manifold with a spin structure that is uniquely determined by the spin structures on the N_i. We assume that N is equipped with such a spin structure which we denote by product spin structure. We proceed by describing the key points of this structure which is thoroughly discussed in [Kli03] (compare also Proposition 1.15 in [LM89]).

The spinor bundle on N can be decomposed into $S(N) = \bigotimes_{i=1}^{l} S_i$. For N_i even-dimensional $S_i \cong S(N_i)$ and for N_i odd-dimensional S_i is either isomorphic to $S(N_i)$ itself or its double. Every section of $S(N)$ can be decomposed into so-called homogeneous sections

$$\sigma = \sigma_1^{\varepsilon_1} \otimes \dots \otimes \sigma_l^{\varepsilon_l}, \qquad \text{where } \sigma_q^{\varepsilon_q} \in \begin{cases} S_q^+, & \varepsilon_q \text{ even} \\ S_q^-, & \varepsilon_q \text{ odd} \end{cases}, \quad q = 1, \dots, l.$$

In case of the product spin structure the Clifford multiplication by a tangent vector $X = X_1 + \ldots + X_l \in TN = TN_1 \times \dots \times TN_l$ is given by

$$c(X) \cdot \sigma_1^{\varepsilon_1} \otimes \dots \otimes \sigma_l^{\varepsilon_l} = \sum_{q=1}^{l} (-1)^{\varepsilon_1 + \ldots + \varepsilon_{q-1}} \sigma_1^{\varepsilon_1} \otimes \dots \otimes c(X_q) \sigma_q^{\varepsilon_q} \otimes \dots \otimes \sigma_l^{\varepsilon_l},$$

(this is Formula (7) on page 3 in [Kli03]).

Since the spin connection $\widetilde{\nabla}^q$ is essentially given by Clifford multiplication by *two* basis sections of TN_q, the signs cancel and we find

$$\widetilde{\nabla}_{e_k}^q \sigma_1^{\varepsilon_1} \otimes \dots \otimes \sigma_l^{\varepsilon_l} = \sigma_1^{\varepsilon_1} \otimes \dots \otimes \nabla_{e_k}^q \sigma_q^{\varepsilon_q} \otimes \dots \otimes \sigma_l^{\varepsilon_l}.$$

For $\widetilde{D}_q,\ q = 1, \dots, l$ this implies

$$\begin{aligned} \widetilde{D}_q \sigma_1^{\varepsilon_1} \otimes \dots \otimes \sigma_l^{\varepsilon_l} &= \sum_{j \in I_q} c(e_j) \widetilde{\nabla}_{e_j}^q \sigma_1^{\varepsilon_1} \otimes \dots \otimes \sigma_l^{\varepsilon_l} \\ &= (-1)^{\varepsilon_1 + \ldots + \varepsilon_{q-1}} \sigma_1^{\varepsilon_1} \otimes \dots \otimes \sum_{j \in I_q} c(e_j) \nabla_{e_j}^q \sigma_q^{\varepsilon_q} \otimes \dots \otimes \sigma_l^{\varepsilon_l} \\ &= (-1)^{\varepsilon_1 + \ldots + \varepsilon_{q-1}} \sigma_1^{\varepsilon_1} \otimes \dots \otimes D_q \sigma_q^{\varepsilon_q} \otimes \dots \otimes \sigma_l^{\varepsilon_l} \end{aligned}$$

and

$$\begin{aligned} \widetilde{D}_p \widetilde{D}_q \sigma_1^{\varepsilon_1} \otimes \dots \otimes \sigma_l^{\varepsilon_l} &= \widetilde{D}_p (-1)^{\varepsilon_1 + \ldots + \varepsilon_{q-1}} \sigma_1^{\varepsilon_1} \otimes \dots \otimes D_q \sigma_q^{\varepsilon_q} \otimes \dots \otimes \sigma_l^{\varepsilon_l} \\ &= \begin{cases} (-1)^{\varepsilon_p + \ldots + \varepsilon_{q-1}} \sigma_1^{\varepsilon_1} \otimes \dots \otimes D_p \sigma_p^{\varepsilon_p} \otimes \dots \otimes D_q \sigma_q^{\varepsilon_q} \otimes \dots \otimes \sigma_l^{\varepsilon_l}, & p < q \\ \sigma_1^{\varepsilon_1} \otimes \dots \otimes D_q^2 \sigma_q^{\varepsilon_q} \otimes \dots \otimes \sigma_l^{\varepsilon_l}, & p = q \\ (-1)^{(\varepsilon_q + 1) + \ldots + \varepsilon_{p-1}} \sigma_1^{\varepsilon_1} \otimes \dots \otimes D_q \sigma_q^{\varepsilon_q} \otimes \dots \otimes D_p \sigma_p^{\varepsilon_p} \otimes \dots \otimes \sigma_l^{\varepsilon_l}, & p > q \end{cases} \end{aligned}$$

since $D_q : S(N_q)^{\pm} \to S(N_q)^{\mp}$.

$$
\begin{aligned}
T(r)^2\sigma_1^{\varepsilon_1}\otimes\cdots\otimes\sigma_l^{\varepsilon_l} &= \sum_{p=1}^{l}\frac{1}{h_p}\widetilde{D}_p\left(\sum_{q=1}^{l}\frac{1}{h_q}\widetilde{D}_q\sigma_1^{\varepsilon_1}\otimes\cdots\otimes\sigma_l^{\varepsilon_l}\right)\\
&= \sum_{q=1}^{l}\frac{1}{h_q^2}\widetilde{D}_q^2\sigma_1^{\varepsilon_1}\otimes\cdots\otimes\sigma_l^{\varepsilon_l} + \sum_{p<q}\frac{1}{h_ph_q}\left(\widetilde{D}_p\widetilde{D}_q+\widetilde{D}_q\widetilde{D}_p\right)\sigma_1^{\varepsilon_1}\otimes\cdots\otimes\sigma_l^{\varepsilon_l}\\
&= \sum_{q=1}^{l}\frac{1}{h_q^2}\sigma_1^{\varepsilon_1}\otimes\cdots\otimes D_q^2\sigma_q^{\varepsilon_q}\otimes\cdots\otimes\sigma_l^{\varepsilon_l} + \sum_{p<q}\frac{1}{h_ph_q}\left((-1)^{\varepsilon_p+\ldots+\varepsilon_{q-1}}+(-1)^{(\varepsilon_p+1)+\ldots+\varepsilon_{q-1}}\right)\\
&\qquad \cdot\sigma_1^{\varepsilon_1}\otimes\cdots\otimes D_p\sigma_p^{\varepsilon_p}\otimes\cdots\otimes D_q\sigma_q^{\varepsilon_q}\otimes\cdots\otimes\sigma_l^{\varepsilon_l}\\
&= \sum_{q=1}^{l}\frac{1}{h_q^2}\sigma_1^{\varepsilon_1}\otimes\cdots\otimes D_q^2\sigma_q^{\varepsilon_q}\otimes\cdots\otimes\sigma_l^{\varepsilon_l}
\end{aligned}
$$

$T(r)$ **is self-adjoint:** $\widetilde{D}_q$ is symmetric on homogeneous sections:

$$
\begin{aligned}
&\left(\widetilde{D}_q\sigma_1^{\varepsilon_1}\otimes\cdots\otimes\sigma_q^{\varepsilon_q}\otimes\cdots\otimes\sigma_l^{\varepsilon_l},\eta_1^{\varepsilon_1}\otimes\cdots\otimes\eta_q^{\varepsilon_q+1}\otimes\cdots\otimes\eta_l^{\varepsilon_l}\right)\\
&\quad=\left((-1)^{\varepsilon_1+\ldots+\varepsilon_{q-1}}\sigma_1^{\varepsilon_1}\otimes\cdots\otimes D_q\sigma_q^{\varepsilon_q}\otimes\cdots\otimes\sigma_l^{\varepsilon_l},\eta_1^{\varepsilon_1}\otimes\cdots\otimes\eta_q^{\varepsilon_q+1}\otimes\cdots\otimes\eta_l^{\varepsilon_l}\right)\\
&\quad=(-1)^{\varepsilon_1+\ldots+\varepsilon_{q-1}}\left(\sigma_1^{\varepsilon_1},\eta_1^{\varepsilon_1}\right)_1\cdots\left(D_q\sigma_q^{\varepsilon_q},\eta_q^{\varepsilon_q+1}\right)_q\cdots\left(\sigma_l^{\varepsilon_l},\eta_l^{\varepsilon_l}\right)_l\\
&\quad=(-1)^{\varepsilon_1+\ldots+\varepsilon_{q-1}}\left(\sigma_1^{\varepsilon_1},\eta_1^{\varepsilon_1}\right)_1\cdots\left(\sigma_q^{\varepsilon_q},D_q\eta_q^{\varepsilon_q+1}\right)_q\cdots\left(\sigma_l^{\varepsilon_l},\eta_l^{\varepsilon_l}\right)_l\\
&\quad=\left(\sigma_1^{\varepsilon_1}\otimes\cdots\otimes\sigma_l^{\varepsilon_l},\widetilde{D}_q\eta_1^{\varepsilon_1}\otimes\cdots\otimes\eta_q^{\varepsilon_q+1}\otimes\cdots\otimes\eta_l^{\varepsilon_l}\right).
\end{aligned}
$$

This implies the symmetry of $T(r)$ since all the other pairings of homogeneous elements vanish. The principal symbol is

$$\sigma_\xi(T(r)) = ic\left(\sum_{q=1}^{l}\frac{1}{h_q}\xi_q\right).$$

Thus, $T(r)$ is a elliptic symmetric differential operator of first order on $S(N)$, where N is a closed manifold. Therefore, the closure $T(r)\in\mathcal{L}(H^1(N,S(N)),L^2(N,S(N)))$, for simplicity denoted with the same letter, is self-adjoint.

The kernel of $T(r)$**:** Every $\sigma\in\Gamma(S(N))$ can be decomposed

$$\sigma=\sum_k\alpha_k\sigma_1^k\otimes\cdots\otimes\sigma_l^k,$$

where σ_q^k is an eigensection of D_q to the eigenvalue λ_q^k and $\alpha_k \neq 0$. If $\sigma \in \ker T(r)$, then

$$\begin{aligned} 0 = T^2(r)\sigma &= \sum_k \alpha_k \left(\sum_{q=1}^{l} \left[\frac{1}{h_q(r)}\right]^2 \sigma_1^k \otimes \cdots \otimes D_q^2 \sigma_q^k \otimes \cdots \otimes \sigma_l^k \right) \\ &= \sum_k \alpha_k \left(\sum_{q=1}^{l} \left[\frac{1}{h_q(r)}\right]^2 (\lambda_q^k)^2 \right) \sigma_1^k \otimes \cdots \otimes \sigma_l^k. \\ \Rightarrow \qquad 0 &= \sum_{q=1}^{l} \left[\frac{1}{h_q(r)}\right]^2 (\lambda_q^k)^2 \quad \forall k \quad \Longrightarrow \quad \lambda_q^k = 0 \quad \forall\, q,k \end{aligned}$$

Thus, $\sigma \in \ker T(r)$ if and only if $\sigma = \sum_k \alpha_k \sigma_1^k \otimes \cdots \otimes \sigma_l^k$ with $\sigma_q^k \in \ker D_q$ for all q,k. $\square$

Theorem 5.2.5. *Let $M = M_1 \cup U$ be a spin manifold with multiply warped product singularity, equipped with the product spin structure and the corresponding irreducible complex spin bundle $S(M)$. Let $D : S(M)^+ \to S(M)^-$ be the associated Spin-Dirac operator.*

If there is a $K > 0$ and a $\beta > 1$, such that

$$\max_{q=1,\dots,l} h_q(r) \leqslant K r^\beta \qquad \text{for all } r \in (0, s_0) \tag{5.1}$$

and for all $q = 1, \dots, l$

$$\int_0^{s_0} r^\beta \left| \frac{h_q'(r)}{h_q(r)} \right|^2 dr < \infty, \tag{5.2}$$

then

$$\operatorname{ind} D_{\min} = \int_{M_1} \widehat{A} + \frac{1}{2}\left(\eta(D_N) - \prod_{q=1}^{l} \dim\ker D_q \right).$$

The closed extensions of $D_{\min}$ are defined by

$$W \subset \mathcal{D}(D_{\max}) \big/ \mathcal{D}(D_{\min}) \cong \bigotimes_{q=1}^{l} \ker D_q, \quad \mathcal{D}(D_W) := \mathcal{D}(D_{\min}) \oplus W, \quad D_W := D_{\max}|_{\mathcal{D}(D_W)}.$$

The extensions are all Fredholm, and their indices are given by $\operatorname{ind} D_W = \operatorname{ind} D_{\min} + \dim W$.

Proof. We first prove that D is an operator with C^1-horn singularity as defined in Definition 2.1.2, i.e. we have to show that $D|_U$ is unitary equivalent to a horn operator as defined in Definition 2.1.1.

Lemma 5.2.3 states that

$$D|_U \cong \frac{\partial}{\partial r} - \sum_{q=1}^{l} \frac{1}{h_q} \widetilde{D}_q.$$

A comparison with Definition 2.1.1 shows that in this case

$$H = \left(\bigotimes_{q=1}^{l} \ker D_q \right)^{\perp}, \qquad \widetilde{H} = \bigotimes_{q=1}^{l} \ker D_q$$

$$S(r) = - \sum_{q=1}^{l} \frac{r^\beta}{h_q(r)} \widetilde{D}_q \Bigg|_H = -r^\beta T(r)|_H, \quad S_1 \equiv 0, \qquad \widetilde{S} = 0 \text{ and } \widetilde{S}_1 \equiv 0.$$

It remains to be shown that these operator families satisfy the asserted properties:

1. *Regularity:* By Lemma 5.2.4 the elliptic symmetric first order differential operators $S(r)$ are self-adjoint in $\in \mathcal{L}(H_1, H)$. The family $r \mapsto S(r)$ is strongly continuously differentiable since the warping functions are C^1.

2. *Spectral gap:* If $\sigma \in H$, $\sigma = \sum_k \alpha_k \sigma_1^k \otimes \cdots \otimes \sigma_l^k$ with $D_q \sigma_q^k = \lambda_q^k \sigma_q^k$, $\lambda_q^k \neq 0$,

$$\|S(r)\sigma\|^2 = (S(r)^2\sigma, \sigma) = \sum_k |\alpha_k|^2 \left(\sum_{q=1}^{l} \left[\frac{r^\beta}{h_q(r)} \right]^2 (\lambda_q^k)^2 \right) \left\| \sigma_1^k \otimes \ldots \otimes \sigma_l^k \right\|^2$$

$$\overset{(5.1)}{\geqslant} K^{-2} \min \left\{ \lambda_q^2 \mid \lambda_q \in \operatorname{spec} D_q \setminus \{0\}, \; q = 1 \ldots, l \right\} \|\sigma\|^2 =: C_2^2 \|\sigma\|^2 .$$

   This implies $|S(r)| \geqslant C_2$.

3. *Spectral projections:* The family

$$S'(r)S(s_0)^{-1} = - \sum_{q=1}^{l} \left(\frac{\beta}{r} - \frac{h_q'(r)}{h_q(r)} \right) \frac{r^\beta}{h_q(r)} \widetilde{D}_q \Bigg|_H \left(-s_0^\beta \sum_{q=1}^{l} \widetilde{D}_q \Bigg|_H \right)^{-1}$$

   is continuous in norm on $I = (0, s_0]$ since the warping functions h_q are C^1. Therefore, the asserted properties follow with Lemma 2.2.2.

4. *Variation:*

$$\alpha(r) = \left\| S'(r)S(r)^{-1} \right\| = \left\| - \sum_{q=1}^{l} \left(\frac{\beta}{r} - \frac{h_q'(r)}{h_q(r)} \right) \frac{r^\beta}{h_q(r)} \widetilde{D}_q \Bigg|_H \left(- \sum_{q=1}^{l} \frac{r^\beta}{h_q(r)} \widetilde{D}_q \Bigg|_H \right)^{-1} \right\|$$

$$\leqslant \sum_{q=1}^{l} \left[\frac{\beta}{r} + \left| \frac{h_q'(r)}{h_q(r)} \right| \right]$$

   Thus, $\beta > 1$ and assumption (5.2) imply

$$\int_0^{s_0} r^\beta \alpha^2(r) dr \leqslant 2l \sum_{q=1}^{l} \int_0^{s_0} r^\beta \left[\frac{\beta^2}{r^2} + \left| \frac{h_q'(r)}{h_q(r)} \right|^2 \right] dr < \infty.$$

Assertion 5., 6. and 7. are trivially satisfied since $S_1 \equiv 0$, $\widetilde{S} = 0$ and $\widetilde{S}_1 \equiv 0$.

In summary, D is an operator with C^1-horn singularity, and Theorem 2.1.3 can be applied. We compute the contents of the index formula:

$$S_0 = S(s_0) = -\sum_{q=1}^{l} \frac{s_0^\beta}{h_q(s_0)} \widetilde{D}_q = -s_0^\beta \sum_{q=1}^{l} \widetilde{D}_q = -s_0^\beta D_N$$

$$\ker S_0 = \bigotimes_{q=1}^{l} \ker D_q \quad \text{and} \quad \dim\ker S_0 = \prod_{q=1}^{l} \dim\ker D_q.$$

By Section 4 in [APS75] page 60f. which refers to the article [ABP73] (note also [ABP75]), $\operatorname{Res}_k \eta_{D_N} = 0$ for all $k \geqslant 1$. □

Examples:

1. *Manifolds with multiple metric horns:* Let $h_q \in C^1((0,s_0],(0,\infty))$ with $h_q(r) = r^{\beta_q}$, $\beta_q > 1$ for r close to 0 for all $q \in \{1,\dots,l\}$. Then assertion (5.1) and (5.2) are satisfied:

$$\max_{q=1,\dots,l} h_q(r) \leqslant K r^\beta, \qquad \text{where } \beta = \min_{q\in\{1,\dots,l\}} \beta_q > 1$$

$$\int_0^{s_0} r^\beta \left|\frac{h_q'(r)}{h_q(r)}\right|^2 dr \leqslant C \int_0^{s_0} r^\beta \left|\frac{\beta_q}{r}\right|^2 dr = C\beta_q^2 \int_0^{s_0} r^{\beta-2} dr \overset{\beta-2>-1}{<} \infty,$$

   for $q = 1,\dots,l$. Thus, Theorem 5.2.5 applies with no more assumptions than $\beta_q > 1$ for all $q \in \{1,\dots,l\}$.

2. *Manifolds with metric horns:* This is a special case of the first example: $l = 1$ and $h_1(r) = r^\beta$ close to zero. This is Theorem 5.3 in [LP98].

5.3 The index of the Gauss-Bonnet and the Signature operator

In this section we look at the Dirac operator $D = d + \delta$ on the bundle of differential forms $\Omega(M)$, where M is a manifold with multiply warped product singularity. We compute the index of the Gauss-Bonnet operator $D^{GB} = D|_{\Omega^{ev}(M)}$ and for $\dim M = 4k$ the index of the Signature operator $D_S = D|_{\Omega^+(M)}$. In the first step, we move the warping functions from the metric to the operator.

Lemma 5.3.1. *Consider the unitary map*

$$\Phi : L^2\left(U,\Lambda(U), g = dr^2 \oplus h_1^2(r)g_1 \oplus \cdots \oplus h_l^2(r)g_l\right) \to L^2\left(U,\Lambda(U), dr^2 \oplus g_1 \oplus \cdots \oplus g_l\right)$$
$$\Phi(e^I) = \Phi(h_I dx_I) := h^{\frac{1}{2}} dx_I \text{ and } \Phi(dr\wedge e^I) := h^{\frac{1}{2}} dr \wedge dx_I \text{ for all } I \subset \{1,\dots,n\}.$$

The operator $D := d + \delta : \Omega(M) \to \Omega(M)$ satisfies

$$\Phi\left(-c(\partial_r) D|_U\right)\Phi^{-1} = \frac{\partial}{\partial r} - c(\partial_r)\widetilde{D}_N + \widetilde{A}(r),$$

where

$$\begin{aligned}
c(\partial_r) &:= w(dr) - \iota\left(\frac{\partial}{\partial r}\right), \quad c(\partial_r)^2 = -I, \\
\widetilde{D}_N f dx_I &:= \sum_{q=1}^{l} \frac{1}{h_q} \sum_{i \in I_q} \frac{\partial f}{\partial x_i} \left(w(dx_i) - \iota\left(\frac{\partial}{\partial x_i}\right)\right) dx_I = \sum_{q=1}^{l} \frac{1}{h_q} \widetilde{D}_q f dx_I \\
&= \left(\widetilde{d}_N + \widetilde{\delta}_N\right) f dx_I, \\
\widetilde{D}_q(f dx_I) &:= (-1)^{|I \cap (\{0\} \cup I_1 \cup \cdots \cup I_{q-1})|} dx_{I \cap (\{0\} \cup I_1 \cup \cdots \cup I_{q-1})} \wedge D_q\left(f dx_{I \cap I_q}\right) \wedge dx_{I \cap (I_{q+1} \cup \cdots \cup I_l)},
\end{aligned}$$

where $D_q := d_q + \delta_q$ *on* $\Omega(N_q)$*, and for all* $I \subset \{1, \ldots, n\}$ *and* $f, g \in C^\infty((0, s_0], \mathbb{R})$

$$\widetilde{A}(r)(f dx_I + g dr \wedge dx_I) := \sum_{q=1}^{l} \frac{h_q'(r)}{h_q(r)} \left(|I \cap I_q| - \frac{n_q}{2}\right) (f dx_I - g dr \wedge dx_I).$$

Proof. This lemma is proved by some straightforward (but tedious) computations. We start off with some auxiliary calculations.

$$\begin{aligned}
\frac{\partial h^{-\frac{1}{2}} h_I}{\partial r} &= \frac{\partial}{\partial r}\left[\prod_{p=1}^{l} h_p^{-\frac{n_p}{2} + |I \cap I_p|}\right] = \sum_{q=1}^{l} \left(-\frac{n_q}{2} + |I \cap I_q|\right) h_q^{-\frac{n_q}{2} + |I \cap I_q| - 1} h_q' \prod_{\substack{p=1 \\ p \neq q}}^{l} h_p^{-\frac{n_p}{2} + |I \cap I_p|} \\
&= \left[\sum_{q=1}^{l} \left(-\frac{n_q}{2} + |I \cap I_q|\right) \frac{h_q'}{h_q}\right] h^{-\frac{1}{2}} h_I
\end{aligned}$$

Let $J \subset I_q$ for a $q \in \{1, \ldots, l\}$ and $f \in C^\infty((0, s_0], \mathbb{R})$.

$$\begin{aligned}
d_q(f dx_J) &= \sum_{i \in I_q \backslash J} \frac{\partial f}{\partial x_i} dx_i \wedge dx_J \\
\delta_q(f dx_J) &= (-1)^{n_q|J| + n_q + 1} *_q d_q\left(f \operatorname{sgn}(J, I_q \backslash J) dx_{I_q \backslash J}\right) \\
&= (-1)^{n_q|J| + n_q + 1} \operatorname{sgn}(J, I_q \backslash J) *_q \left(\sum_{i \in J} \frac{\partial f}{\partial x_i} dx_i \wedge dx_{I_q \backslash J}\right) \\
&= (-1)^{n_q|J| + n_q + 1} \operatorname{sgn}(J, I_q \backslash J) \sum_{i \in J} \operatorname{sgn}(i, I_q \backslash J, J \backslash \{i\}) \frac{\partial f}{\partial x_i} dx_{J \backslash \{i\}} \\
&= (-1)^{n_q|J| + n_q + 1} (-1)^{|J|(n_q - |J|)} (-1)^{n_q - |J|} \sum_{i \in J} \operatorname{sgn}(i, J \backslash \{i\}) \frac{\partial f}{\partial x_i} dx_{J \backslash \{i\}} \\
&= -\sum_{i \in J} \operatorname{sgn}(i, J \backslash \{i\}) \frac{\partial f}{\partial x_i} dx_{J \backslash \{i\}}
\end{aligned}$$

We abbreviate $\sim\! I := \{1, \ldots, n\} \backslash I$.

$$\Phi d \Phi^{-1}(f dx_I) = \Phi d\left(f h^{-\frac{1}{2}} e^I\right) = \Phi d\left(f h^{-\frac{1}{2}} h_I dx_I\right)$$

$$
\begin{aligned}
&= \Phi\left[\left(\frac{\partial f}{\partial r} h^{-\frac{1}{2}} h_I + f \frac{\partial h^{-\frac{1}{2}} h_I}{\partial r} \right) dr \wedge dx_I + h^{-\frac{1}{2}} h_I \sum_{i \in \sim I} \frac{\partial f}{\partial x_i} dx_i \wedge dx_I \right] \\
&= \left(\frac{\partial f}{\partial r} + f \frac{\partial \log \left(h^{-\frac{1}{2}} h_I \right)}{\partial r} \right) dr \wedge dx_I + \sum_{i \in \sim I} \frac{\partial f}{\partial x_i} \frac{1}{h_{q(i)}} dx_i \wedge dx_I \\
&= \left(\frac{\partial f}{\partial r} + f \sum_{q=1}^{l} \left(|I \cap I_q| - \frac{n_q}{2} \right) \frac{h_q'}{h_q} \right) dr \wedge dx_I \\
&\quad + \sum_{q=1}^{l} \frac{1}{h_q} (-1)^{|I \cap (I_1 \cup \cdots \cup I_{q-1})|} dx_{I \cap (I_1 \cup \cdots \cup I_{q-1})} \wedge d_q \left(f dx_{I \cap I_q} \right) \wedge dx_{I \cap (I_{q+1} \cup \cdots \cup I_l)} \\
&=: \left(\frac{\partial f}{\partial r} + f \sum_{q=1}^{l} \left(|I \cap I_q| - \frac{n_q}{2} \right) \frac{h_q'}{h_q} \right) dr \wedge dx_I + \sum_{q=1}^{l} \frac{1}{h_q} \widetilde{d}_q \left(f dx_I \right).
\end{aligned}
$$

$$
\begin{aligned}
&\Phi d \Phi^{-1}(f dr \wedge dx_I) = \Phi d \left(f h^{-\frac{1}{2}} h_I dr \wedge dx_I \right) \\
&\quad = \Phi \left[\sum_{i \in \sim I} \frac{\partial f}{\partial x_i} h^{-\frac{1}{2}} h_I dx_i \wedge dr \wedge dx_I \right] = \sum_{i \in \sim I} \frac{\partial f}{\partial x_i} \frac{1}{h_{q(i)}} dx_i \wedge dr \wedge dx_I \\
&\quad = \sum_{q=1}^{l} \frac{1}{h_q} (-1)^{|I \cap (I_1 \cup \cdots \cup I_{q-1})|+1} dr \wedge dx_{I \cap (I_1 \cup \cdots \cup I_{q-1})} \wedge d_q \left(f dx_{I \cap I_q} \right) \wedge dx_{I \cap (I_{q+1} \cup \cdots \cup I_l)} \\
&\quad =: \sum_{q=1}^{l} \frac{1}{h_q} \widetilde{d}_q \left(f dr \wedge dx_I \right)
\end{aligned}
$$

$$
\begin{aligned}
&\Phi \delta \Phi^{-1}(f dx_I) = (-1)^{(n+1)|I|+n} \Phi * d * (f h^{-\frac{1}{2}} e^I) \\
&\qquad = (-1)^{(n+1)|I|+n} \operatorname{sgn}(I, 0, \sim I) \Phi * d(f h^{-\frac{1}{2}} dr \wedge h_{\sim I} dx_{\sim I}) \\
&\qquad = (-1)^{(n+1)|I|+n} \operatorname{sgn}(I, 0, \sim I) \Phi * \left[\sum_{i \in I} \frac{1}{h_{q(i)}} \frac{\partial f}{\partial x_i} h^{-\frac{1}{2}} h_{q(i)} dx_i \wedge dr \wedge h_{\sim I} dx_{\sim I} \right] \\
&\qquad = (-1)^{(n+1)|I|+n} \operatorname{sgn}(I, 0, \sim I) \left[\sum_{i \in I} \operatorname{sgn}(i, 0, \sim I, I \backslash \{i\}) \frac{1}{h_{q(i)}} \frac{\partial f}{\partial x_i} dx_{I \backslash \{i\}} \right] \\
&\qquad = - \sum_{i \in I} \operatorname{sgn}(i, I \backslash \{i\}) \frac{1}{h_{q(i)}} \frac{\partial f}{\partial x_i} dx_{I \backslash \{i\}} \\
&\qquad = \sum_{q=1}^{l} \frac{1}{h_q} (-1)^{|I \cap (I_1 \cup \cdots \cup I_{q-1})|} dx_{I \cap (I_1 \cup \cdots \cup I_{q-1})} \wedge \delta_q \left(f dx_{I \cap I_q} \right) \wedge dx_{I \cap (I_{q+1} \cup \cdots \cup I_l)} \\
&\qquad =: \sum_{q=1}^{l} \frac{1}{h_q} \widetilde{\delta}_q (f dx_I)
\end{aligned}
$$

$$
\begin{aligned}
&\Phi\delta\Phi^{-1}(fdr\wedge dx_I) = (-1)^{(n+1)(|I|+1)+n}\Phi *d* \left[h^{-\frac{1}{2}} f dr\wedge e^I\right] \\
&\quad = (-1)^{(n+1)|I|+1}\operatorname{sgn}(0,I,\sim I)\Phi *d\left[fh^{-\frac{1}{2}}h_{\sim I}dx_{\sim I}\right] \\
&\quad = (-1)^{(n+1)|I|+1}\operatorname{sgn}(0,I,\sim I)\Phi *\left[\left(\frac{\partial f}{\partial r} + f\frac{\partial \log h^{-\frac{1}{2}}h_{\sim I}}{\partial r}\right)h^{-\frac{1}{2}}dr\wedge h_{\sim I}dx_{\sim I}\right. \\
&\qquad \left. + \sum_{i\in I}\frac{1}{h_{q(i)}}\frac{\partial f}{\partial x_i}h^{-\frac{1}{2}}h_{q(i)}dx_i\wedge h_{\sim I}dx_{\sim I}\right] \\
&\quad = (-1)^{(n+1)|I|+1}\operatorname{sgn}(0,I,\sim I)\left[\left(\frac{\partial f}{\partial r} + f\frac{\partial \log h^{-\frac{1}{2}}h_{\sim I}}{\partial r}\right)\operatorname{sgn}(0,\sim I,I)dx_I\right. \\
&\qquad \left. + \sum_{i\in I}\frac{1}{h_{q(i)}}\frac{\partial f}{\partial x_i}\operatorname{sgn}(i,\sim I,0,I\setminus\{i\})dr\wedge dx_{I\setminus\{i\}}\right] \\
&\quad = -\left(\frac{\partial f}{\partial r} + f\frac{\partial \log h^{-\frac{1}{2}}h_{\sim I}}{\partial r}\right)dx_I + \sum_{i\in I}\frac{1}{h_{q(i)}}\frac{\partial f}{\partial x_i}\operatorname{sgn}(i,I\setminus\{i\})dr\wedge dx_{I\setminus\{i\}} \\
&\quad = -\left(\frac{\partial f}{\partial r} + f\sum_{q=1}^{l}\left(|\sim I\cap I_q| - \frac{n_q}{2}\right)\frac{h_q'}{h_q}\right)dx_I \\
&\qquad + \sum_{q=1}^{l}\frac{1}{h_q}(-1)^{|I\cap(I_1\cup\cdots\cup I_{q-1})|+1}dr\wedge dx_{I\cap(I_1\cup\cdots\cup I_{q-1})}\wedge\delta_q\left(fdx_{I\cap I_q}\right)\wedge dx_{I\cap(I_{q+1}\cup\cdots\cup I_l)} \\
&\quad =: -\frac{\partial f}{\partial r}dx_I + f\sum_{q=1}^{l}\frac{h_q'}{h_q}\left(|I\cap I_q| - \frac{n_q}{2}\right)dx_I + \sum_{q=1}^{l}\frac{1}{h_q}\widetilde{\delta}_q(fdr\wedge dx_I)
\end{aligned}
$$

Summing up, gives

$$
\begin{aligned}
\Phi(d+\delta)\Phi^{-1}(fdx_I) &= \left(\frac{\partial f}{\partial r} + f\sum_{q=1}^{l}\frac{h_q'}{h_q}\left(|I\cap I_q| - \frac{n_q}{2}\right)\right)dr\wedge dx_I \\
&\quad + \sum_{q=1}^{l}\frac{1}{h_q}\left(\widetilde{d}_q + \widetilde{\delta}_q\right)(fdx_I) \\
&= c(\partial_r)\left[\left(\frac{\partial f}{\partial r} + f\sum_{q=1}^{l}\frac{h_q'}{h_q}\left(|I\cap I_q| - \frac{n_q}{2}\right)\right)dx_I\right. \\
&\quad \left. - c(\partial_r)\sum_{q=1}^{l}\frac{1}{h_q}\left(\widetilde{d}_q + \widetilde{\delta}_q\right)(fdx_I)\right]
\end{aligned}
$$

$$\begin{aligned}\Phi(d+\delta)\Phi^{-1}(fdr\wedge dx_I) &= -\frac{\partial f}{\partial r}dx_I + f\sum_{q=1}^{l}\frac{h_q'}{h_q}\left(|I\cap I_q| - \frac{n_q}{2}\right)dx_I \\ &+\sum_{q=1}^{l}\frac{1}{h_q}\left(\widetilde{d}_q+\widetilde{\delta}_q\right)(fdr\wedge dx_I) \\ &= c(\partial_r)\Bigg[\left(\frac{\partial f}{\partial r} - f\sum_{q=1}^{l}\frac{h_q'}{h_q}\left(|I\cap I_q| - \frac{n_q}{2}\right)\right)dr\wedge dx_I \\ &- c(\partial_r)\sum_{q=1}^{l}\frac{1}{h_q}\left(\widetilde{d}_q+\widetilde{\delta}_q\right)(fdr\wedge dx_I)\Bigg].\end{aligned}$$

$\square$

Lemma 5.3.2. *The Gauss-Bonnet and Signature involution*

$$\tau_{GB}\alpha_j := (-1)^j\alpha_j \quad \text{and} \quad \tau_s\alpha_j := (-1)^{k+\frac{j(j-1)}{2}} * \alpha_j, \quad \text{for } \alpha_j\in\Omega^j(U),$$

anti-commute with D, $c(\partial_r)$ and $\widetilde{D}_N$.

Proof. For the Gauss-Bonnet involution the assertions are obviously satisfied. For the Signature involution these computations are straightforward. They are presented here for the sake of completeness. Note that we assumed $\dim M = 4k$ in the Signature case.

1. *τ_s anti-commutes with $D = d+\delta$:* Let $\alpha_j\in\Omega^j(M)$.

$$\begin{aligned}D\tau_s\alpha_j &= \left(d + (-1)^{4k(4k-j)+1} * d *\right)(-1)^{k+\frac{j(j-1)}{2}} * \alpha_j \\ &= (-1)^{k+\frac{j(j-1)}{2}}\left((-1)^{(4k-j+1)(j-1)} ** d * \alpha_j + (-1)^{4k(4k-j)+1} * d(-1)^{j(4k-j)}\alpha_j\right) \\ &= -(-1)^{k+\frac{(j-1)(j-2)}{2}} * \delta\alpha_j - (-1)^{k+\frac{(j+1)j}{2}} * d\alpha_j = -\tau_s D\alpha_j\end{aligned}$$

2. *τ_s anti-commutes with $c(\partial_r)$:* Let $I\subset\{1,\dots,n\}$ with $|I| = j$.

$$\begin{aligned}\tau_s c(\partial_r)dx_I &= \tau_s(dr\wedge dx_I) = (-1)^{k+\frac{(j+1)j}{2}} * dr\wedge dx_I = (-1)^{k+\frac{(j+1)j}{2}}\operatorname{sgn}(0,I,\sim I)dx_{\sim I} \\ c(\partial_r)\tau_s dx_I &= c(\partial_r)(-1)^{k+\frac{j(j-1)}{2}}\operatorname{sgn}(I,0,\sim I)dr\wedge dx_{\sim I} \\ &= (-1)^{k+\frac{j(j-1)}{2}+j+1}\operatorname{sgn}(0,I,\sim I)dx_{\sim I} = -(-1)^{k+\frac{(j+1)j}{2}}\operatorname{sgn}(0,I,\sim I)dx_{\sim I} \\ &\Rightarrow \qquad (\tau_s c(\partial_r) + c(\partial_r)\tau_s)\,dx_I = 0\end{aligned}$$

$$\begin{aligned}c(\partial_r)\tau_s dr\wedge dx_I &= c(\partial_r)\tau_s c(\partial_r)dx_I = -c(\partial_r)^2\tau_s dx_I = \tau_s dx_I \\ &= -\tau_s c(\partial_r)^2 dx_I = -\tau_s c(\partial_r)dr\wedge dx_I \\ &\Rightarrow \qquad (\tau_s c(\partial_r) + c(\partial_r)\tau_s)\,dr\wedge dx_I = 0\end{aligned}$$

3. τ_s *anti-commutes with* $\widetilde{D}_N$: Let $I \subset \{1, \dots, n\}$ with $|I| = j$.

$$\begin{aligned}
\widetilde{d}_N \tau_s f dx_I &= \widetilde{d}_N (-1)^{k+\frac{j(j-1)}{2}} \operatorname{sgn}(I, 0, \sim I) f dr \wedge dx_{\sim I} \\
&= (-1)^{k+\frac{j(j-1)}{2}} \operatorname{sgn}(I, 0, \sim I) \sum_{i \in I} \frac{1}{h_{q(i)}} \frac{\partial f}{\partial x_i} dx_i \wedge dr \wedge dx_{\sim I} \\
&= (-1)^{k+\frac{j(j-1)}{2}} \operatorname{sgn}(I, 0, \sim I) \sum_{i \in I} \frac{1}{h_{q(i)}} \frac{\partial f}{\partial x_i} \operatorname{sgn}(I \backslash \{i\}, i, 0, \sim I) * dx_{I \backslash \{i\}} \\
&= (-1)^{k+\frac{j(j-1)}{2}} \sum_{i \in I} \frac{1}{h_{q(i)}} \frac{\partial f}{\partial x_i} (-1)^{j-1} \operatorname{sgn}(i, I \backslash \{i\}) (-1)^{k+\frac{(j-1)(j-2)}{2}} \tau_s dx_{I \backslash \{i\}} \\
&= \tau_s \left(\sum_{i \in I} \frac{1}{h_{q(i)}} \frac{\partial f}{\partial x_i} \operatorname{sgn}(i, I \backslash \{i\}) dx_{I \backslash \{i\}} \right) = -\tau_s \widetilde{\delta}_N f dx_I
\end{aligned}$$

$$\begin{aligned}
&\widetilde{\delta}_N \tau_s f dx_I = \widetilde{\delta}_N (-1)^{k+\frac{j(j-1)}{2}} \operatorname{sgn}(I, 0, \sim I) f dr \wedge dx_{\sim I} \\
&= (-1)^{k+\frac{j(j-1)}{2}} \operatorname{sgn}(I, 0, \sim I) \sum_{i \in \sim I} \frac{1}{h_{q(i)}} \frac{\partial f}{\partial x_i} \operatorname{sgn}(i, \sim I \backslash \{i\}) dr \wedge dx_{\sim I \backslash \{i\}} \\
&= (-1)^{k+\frac{j(j-1)}{2}} \operatorname{sgn}(I, 0, \sim I) \sum_{i \in \sim I} \frac{1}{h_{q(i)}} \frac{\partial f}{\partial x_i} \operatorname{sgn}(i, \sim I \backslash \{i\}) \operatorname{sgn}(i, I, 0, \sim I \backslash \{i\}) * dx_i \wedge dx_I \\
&= (-1)^{k+\frac{j(j-1)}{2}} \sum_{i \in \sim I} \frac{1}{h_{q(i)}} \frac{\partial f}{\partial x_i} (-1)^{j+1} (-1)^{k+\frac{(j+1)j}{2}} \tau_s dx_i \wedge dx_I \\
&= \tau_s \left(-\sum_{i \in \sim I} \frac{1}{h_{q(i)}} \frac{\partial f}{\partial x_i} dx_i \wedge dx_I \right) = -\tau_s \widetilde{d}_N f dx_I
\end{aligned}$$

$$\Rightarrow \qquad \left(\tau_s \widetilde{D}_N + \widetilde{D}_N \tau_s \right) dx_I = 0$$

$$\begin{aligned}
\tau_s \widetilde{D}_N dr \wedge dx_I &= \tau_s \widetilde{D}_N c(\partial_r) dx_I = -\tau_s c(\partial_r) \widetilde{D}_N dx_I = c(\partial_r) \tau_s \widetilde{D}_N dx_I \\
&= -c(\partial_r) \widetilde{D}_N \tau_s dx_I = -\widetilde{D}_N \tau_s c(\partial_r) dx_I = -\widetilde{D}_N \tau_s dr \wedge dx_I
\end{aligned}$$

$$\Rightarrow \qquad \left(\tau_s \widetilde{D}_N + \widetilde{D}_N \tau_s \right) dr \wedge dx_I = 0$$

□

Lemma 5.3.3. *Under the unitary map*

$$\begin{aligned}
\Psi_{GB} : L^2((0, s_0], L^2(\Lambda(N, g_1 \oplus \cdots \oplus g_l))) &\rightarrow L^2(\Lambda^{ev}(U, dr^2 \oplus g_1 \oplus \cdots \oplus g_l)) \\
\omega(r) = \omega^{ev}(r) + \omega^{odd}(r) &\mapsto \omega^{ev}(r) + dr \wedge \omega^{odd}(r)
\end{aligned}$$

the operators transform into

$$T^{GB}(r) := \Psi_{GB}^{-1}\left(c(\partial_r)\widetilde{D}_N\right)\Psi_{GB} = \sum_{q=1}^{l}\frac{1}{h_q}\widetilde{D}_q \qquad \text{and} \tag{5.3}$$

$$A^{GB}(r)dx_I := \Psi_{GB}^{-1}\widetilde{A}(r)\Psi_{GB}dx_I = \sum_{q=1}^{l}\frac{h_q'}{h_q}\left(|I\cap I_q| - \frac{n_q}{2}\right)(-1)^{|I|}dx_I. \tag{5.4}$$

$T^{GB}(r)$ is a family of elliptic symmetric first order differential operators on $\Omega(N)$ with the property

$$T^{GB}(r)^2\omega_1\wedge\ldots\wedge\omega_l = \sum_{q=1}^{l}\frac{1}{h_q^2}\omega_1\wedge\ldots\wedge\Delta_q\omega_q\wedge\ldots\wedge\omega_l. \tag{5.5}$$

The operators $T^{GB}(r)$ are self-adjoint in $\mathcal{L}(H^1(N,\Lambda(N)),L^2(N,\Lambda(N)))$, where TN is equipped with the metric $g_1\oplus\cdots\oplus g_l$.

$$\ker T^{GB}(r) = \mathcal{H}(N_1)\wedge\cdots\wedge\mathcal{H}(N_l) = \mathcal{H}(N),$$

where $\mathcal{H}(N_q)$ are the harmonic forms on N_q.

Proof. Let $I\subset\{1,\ldots,n\}$ with $|I|$ even:

$$\begin{aligned}A^{GB}(r)dx_I &= \Psi_{GB}^{-1}\widetilde{A}(r)\Psi_{GB}dx_I = \Psi_{GB}^{-1}\widetilde{A}(r)dx_I = \Psi_{GB}^{-1}\sum_{q=1}^{l}\frac{h_q'}{h_q}\left(|I\cap I_q| - \frac{n_q}{2}\right)dx_I\\ &= \sum_{q=1}^{l}\frac{h_q'}{h_q}\left(|I\cap I_q| - \frac{n_q}{2}\right)dx_I.\end{aligned}$$

Let $I\subset\{1,\ldots,n\}$ with $|I|$ odd:

$$\begin{aligned}A^{GB}(r)dx_I &= \Psi_{GB}^{-1}\widetilde{A}(r)dr\wedge dx_I = -\Psi_{GB}^{-1}\sum_{q=1}^{l}\frac{h_q'}{h_q}\left(|I\cap I_q| - \frac{n_q}{2}\right)dr\wedge dx_I\\ &= -\sum_{q=1}^{l}\frac{h_q'}{h_q}\left(|I\cap I_q| - \frac{n_q}{2}\right)dx_I.\end{aligned}$$

Let $\omega\in\Omega(N)$:

$$\begin{aligned}T^{GB}(r)\omega &= \Psi_{GB}^{-1}\left(c(\partial_r)\widetilde{D}_N\right)\Psi_{GB}\omega = \Psi_{GB}^{-1}\left(c(\partial_r)\widetilde{D}_N\right)(\omega^{ev} + dr\wedge\omega^{odd})\\ &= \Psi_{GB}^{-1}\left(dr\wedge\widetilde{D}_N\omega^{ev} + \widetilde{D}_N\omega^{odd}\right) = \widetilde{D}_N\omega^{ev} + \widetilde{D}_N\omega^{odd} = \widetilde{D}_N\omega.\end{aligned}$$

We proceed by computing the properties of $T^{GB}(r)$. We use the following notation: $\omega_q^{\alpha_q}\in\Omega^{ev}(N_q)$ if α_q is even and $\omega_q^{\alpha_q}\in\Omega^{odd}(N_q)$ if α_q is odd. Every differential form in $\Omega(N)$ can be decomposed into homogeneous forms

$$\omega_1^{\alpha_1}\wedge\ldots\wedge\omega_l^{\alpha_l}$$

and therefore, it suffices to show the properties for such elements.

1. *$T^{GB}(r) = \widetilde{D}_N = \sum_{q=1}^{l} h_q^{-1} \widetilde{D}_q$ is symmetric:*

   $$\left(T^{GB}(r)\omega_1^{\alpha_1} \wedge \ldots \wedge \omega_l^{\alpha_l}, \eta_1^{\alpha_1} \wedge \ldots \wedge \eta_p^{\alpha_p+1} \wedge \ldots \wedge \eta_l^{\alpha_l}\right)$$
   $$= \left(\sum_{q=1}^{l} \frac{1}{h_q(r)} (-1)^{\alpha_1+\ldots+\alpha_{q-1}} \omega_1^{\alpha_1} \wedge \ldots \wedge D_q\omega_q^{\alpha_q} \wedge \ldots \wedge \omega_l^{\alpha_l}, \eta_1^{\alpha_1} \wedge \ldots \wedge \eta_p^{\alpha_p+1} \wedge \ldots \wedge \eta_l^{\alpha_l} \right)$$
   $$= (-1)^{\alpha_1+\ldots+\alpha_{p-1}} (\omega_1^{\alpha_1}, \eta_1^{\alpha_1})_1 \cdots \underbrace{\left(D_p\omega_p^{\alpha_p}, \eta_p^{\alpha_p+1}\right)_p}_{=\left(\omega_p^{\alpha_p}, D_p\eta_p^{\alpha_p+1}\right)} \cdots (\omega_l^{\alpha_l}, \eta_l^{\alpha_l})_l$$
   $$= \left(\omega_1^{\alpha_1} \wedge \ldots \wedge \omega_l^{\alpha_l}, T^{GB}(r)\eta_1^{\alpha_1} \wedge \ldots \wedge \eta_p^{\alpha_p+1} \wedge \ldots \wedge \eta_l^{\alpha_l}\right),$$

   where we have used that $\left(\omega_p^{\alpha_p}, \eta_p^{\alpha_p+1}\right)_p = 0$.

2. *$T^{GB}(r)^2$ is the sum of the Laplace operators over the manifolds N_q:*

   $$T^{GB}(r)^2\omega_1^{\alpha_1} \wedge \ldots \wedge \omega_l^{\alpha_l} = \left(\sum_{q=1}^{l} \frac{1}{h_q}\widetilde{D}_q\right)^2 \omega_1^{\alpha_1} \wedge \ldots \wedge \omega_l^{\alpha_l}$$
   $$= \sum_{p<q} \frac{1}{h_p h_q} \overbrace{\left((-1)^{\alpha_p+\cdots+\alpha_{q-1}} + (-1)^{(\alpha_p+1)+\cdots+\alpha_{q-1}}\right)}^{=0} \cdot$$
   $$\cdot\, \omega_1^{\alpha_1} \wedge \ldots \wedge D_p\omega_p^{\alpha_p} \wedge \ldots \wedge D_q\omega_q^{\alpha_q} \wedge \ldots \wedge \omega_l^{\alpha_l}$$
   $$+ \sum_{q=1}^{l} \frac{1}{h_q^2} \omega_1^{\alpha_1} \wedge \ldots \wedge \Delta_q\omega_q^{\alpha_q} \wedge \ldots \wedge \omega_l^{\alpha_l}.$$

3. *$T^{GB}(r)$ is self-adjoint:* We compute the principal symbol. Let $\phi \in C^\infty(N)$ with $\phi(x) = 0$ and $d\phi(x) = \xi$, and $\omega \in \Omega(N)$ with $\omega(x) = e$.

   $$\left(\widehat{T^{GB}(r)}(\xi)\right)^2[e] = \widehat{T^{GB}(r)^2}(\xi)[e] = i^2 \left(T^{GB}(r)\right)^2 (\phi\omega)(x)$$
   $$= \sum_k \alpha_k \sum_{q=1}^{l} \frac{1}{h_q^2(r)} \omega_1^k \wedge \ldots \wedge i^2\Delta_q(\phi\omega_q^k) \wedge \ldots \wedge \omega_l^k(x)$$
   $$= \sum_k \alpha_k \sum_{q=1}^{l} \frac{1}{h_q^2(r)} \omega_1^k \wedge \ldots \wedge \|\xi_q\|^2 \omega_q^k \wedge \ldots \wedge \omega_l^k(x) = \sum_{q=1}^{l} \left[\frac{\|\xi_q\|}{h_q(r)}\right]^2 e$$

   This implies that $T^{GB}(r)$ is elliptic. Thus, $T^{GB}(r)$ is a family of elliptic symmetric differential operators of first order on $\Omega(N)$, where N is a closed manifold. The closure of the operator also denoted by $T^{GB}(r) \in \mathcal{L}(H^1(N, \Lambda(N)), L^2(N, \Lambda(N)))$ is self-adjoint.

4. *The kernel of $T^{GB}(r)$:* Every differential form $\omega \in \Omega(N)$ can be decomposed into $\omega = \sum_k \alpha_k\omega_1^k \wedge \ldots \wedge \omega_l^k$, where ω_q^k is an eigenform of Δ_q to the eigenvalue λ_q^k and $\alpha_k \neq 0$. If

$\omega \in \ker T^{GB}(r)$, then

$$
\begin{aligned}
0 = T^{GB}(r)^2\omega &= \sum_k \alpha_k \left(\sum_{q=1}^{l} \frac{1}{h_q^2(r)} \omega_1^k \wedge \ldots \wedge \Delta_q \omega_q^k \wedge \ldots \wedge \omega_l^k \right) \\
&= \sum_k \alpha_k \left(\sum_{q=1}^{l} \frac{1}{h_q^2(r)} \lambda_q^k \right) \omega_1^k \wedge \ldots \wedge \omega_l^k . \\
\Rightarrow \qquad 0 &= \sum_{q=1}^{l} \frac{1}{h_q^2(r)} \lambda_q^k \quad \forall k \quad \stackrel{\lambda_q^k \geqslant 0}{\Longrightarrow} \quad \lambda_q^k = 0 \quad \forall\, q, k
\end{aligned}
$$

Thus, $\omega \in \ker T^{GB}(r)$ if and only if $\omega = \sum_k \alpha_k \omega_1^k \wedge \ldots \wedge \omega_l^k$ with $\omega_q^k \in \ker \Delta_q = \mathcal{H}(N_q)$ for all q, k, i.e.

$$
\ker T^{GB}(r) = \ker T^{GB}(r)^2 = \mathcal{H}(N_1) \wedge \cdots \wedge \mathcal{H}(N_l) = \mathcal{H}(N).
$$

□

Lemma 5.3.4. *Under the unitary map*

$$
\begin{aligned}
\Psi_s : L^2((0, s_0], L^2(\Lambda(N, g_1 \oplus \cdots \oplus g_l))) &\rightarrow L^2(\Lambda^+(U, dr^2 \oplus g_1 \oplus \cdots \oplus g_l)) \\
\omega(r) &\mapsto \frac{1}{2}(1 + \tau_s)\omega(r)
\end{aligned}
$$

the operators transform into

$$
T^s(r) := \Psi_s^{-1} \left(c(\partial_r) \widetilde{D}_N \right) \Psi_s = \tau' \widetilde{D}_N = \widetilde{D}_N \tau'
$$

where $\tau' \alpha_j = (-1)^{k + \frac{(j+1)j}{2}} *_N \alpha_j$ *for* $\alpha_j \in \Omega^j(N)$ *and*

$$
A^s(r) dx_I := \Psi_s^{-1} A(r) \Psi_s dx_I = \sum_{q=1}^{l} \frac{h_q'}{h_q} \left(|I \cap I_q| - \frac{n_q}{2} \right) dx_I .
$$

$T^s(r)$ is a family of elliptic symmetric first order differential operators on $\Omega(N)$ with the property

$$
T^s(r)^2 \omega_1 \wedge \ldots \wedge \omega_l = \sum_{q=1}^{l} \frac{1}{h_q^2} \omega_1 \wedge \ldots \wedge \Delta_q \omega_q \wedge \ldots \wedge \omega_l .
$$

The operators $T^s(r)$ are self-adjoint in $\mathcal{L}(H^1(N, \Lambda(N)), L^2(N, \Lambda(N)))$, where TN is equipped with the metric $g_1 \oplus \cdots \oplus g_l$, and $\ker T^s(r) = \mathcal{H}(N_1) \wedge \cdots \wedge \mathcal{H}(N_l) = \mathcal{H}(N)$.

Proof. We start off by proving that Ψ_s is an isomorphism. Let $\omega, \eta \in \Omega(N)$ and

$$
\Psi_s'(\omega + dr \wedge \eta) := \omega + \tau_s dr \wedge \eta .
$$

On the one hand,

$$\Psi_s'\Psi_s\omega = \Psi_s'\frac{1}{2}(\omega + \tau_s\omega) = \frac{1}{2}\omega + \tau_s^2\frac{1}{2}\omega = \omega.$$

On the other hand,

$$\begin{aligned}
\Psi_s\Psi_s'\frac{1}{2}(1+\tau_s)(\omega + dr\wedge\eta) &= \Psi_s\frac{1}{2}\Psi_s'(\omega + \tau_s dr\wedge\eta + \tau_s\omega + dr\wedge\eta)\\
&= \Psi_s\frac{1}{2}(\omega + \tau_s dr\wedge\eta + \tau_s^2\omega + \tau_s dr\wedge\eta) = \frac{1}{2}(1+\tau_s)(\omega + \tau_s dr\wedge\eta)\\
&= \frac{1}{2}(\omega + \tau_s dr\wedge\eta + \tau_s\omega + dr\wedge\eta) = \frac{1}{2}(1+\tau_s)(\omega + dr\wedge\eta).
\end{aligned}$$

Next, we compute $A^s(r)$.

$$\begin{aligned}
A^s(r)dx_I &= \Psi_s^{-1}\widetilde{A}(r)\Psi_s dx_I = \Psi_s^{-1}\frac{1}{2}\widetilde{A}(r)(dx_I + \tau_s dx_I)\\
&= \Psi_s^{-1}\frac{1}{2}\widetilde{A}(r)(dx_I + (-1)^{k+\frac{j(j-1)}{2}}\operatorname{sgn}(I,0,\sim I)dr\wedge dx_{\sim I})\\
&= \Psi_s^{-1}\frac{1}{2}\Bigg(\sum_{q=1}^{l}\frac{h_q'}{h_q}\left(|I\cap I_q| - \frac{n_q}{2}\right)dx_I\\
&\qquad - (-1)^{k+\frac{j(j-1)}{2}}\operatorname{sgn}(I,0,\sim I)\sum_{q=1}^{l}\frac{h_q'}{h_q}\Big(\underbrace{|\sim I\cap I_q|}_{=n_q-|I\cap I_q|} - \frac{n_q}{2}\Big)dr\wedge dx_{\sim I}\Bigg)\\
&= \Psi_s^{-1}\sum_{q=1}^{l}\frac{h_q'}{h_q}\left(|I\cap I_q| - \frac{n_q}{2}\right)\frac{1}{2}(1+\tau_s)dx_I = \sum_{q=1}^{l}\frac{h_q'}{h_q}\left(|I\cap I_q| - \frac{n_q}{2}\right)dx_I
\end{aligned}$$

τ' commutes with $\widetilde{D}_N$: Let $I\subset\{1,\dots,n\}$ with $|I| = j$.

$$\begin{aligned}
\tau_s c(\partial_r)dx_I &= \tau_s dr\wedge dx_I = (-1)^{k+\frac{(j+1)j}{2}}\operatorname{sgn}(0,I,\sim I)dx_{\sim I} = (-1)^{k+\frac{(j+1)j}{2}} *_N dx_I = \tau' dx_I\\
\widetilde{D}_N\tau' dx_I &= \widetilde{D}_N\tau_s c(\partial_r)dx_I = -\tau_s\widetilde{D}_N c(\partial_r)dx_I = \tau_s c(\partial_r)\widetilde{D}_N dx_I = \tau'\widetilde{D}_N dx_I
\end{aligned}$$

We compute $T^s(r)$: Let $\alpha_j\in\Omega^j(N)$.

$$T^s(r)\alpha_j = \Psi_s^{-1}c(\partial_r)\widetilde{D}_N\frac{1+\tau_s}{2}\alpha_j = \Psi_s^{-1}\frac{1+\tau_s}{2}\left(dr\wedge\widetilde{D}_N\alpha_j\right) = \tau_s dr\wedge\widetilde{D}_N\alpha_j = \tau'\widetilde{D}_N\alpha_j$$

$T^s(r)$ is symmetric: Let $\omega,\eta\in\Omega(N)$.

$$\left(\tau'\widetilde{D}_N\omega,\eta\right) = \left(\widetilde{D}_N\omega,\tau'\eta\right) = \left(\omega,\widetilde{D}_N\tau'\eta\right) = \left(\omega,\tau'\widetilde{D}_N\eta\right)$$

The symmetry of $\widetilde{D}_N$ has been shown in Lemma 5.3.3. The square

$$T^s(r)^2 = \tau'\widetilde{D}_N\tau'\widetilde{D}_N = \tau'^2\widetilde{D}_N^2 = \widetilde{D}_N^2 = T^{GB}(r)^2.$$

Therefore, the rest of the proof is the same as in Lemma 5.3.3. □

Theorem 5.3.5. *Let $M = M_1 \cup U$ be a manifold with multiply warped product singularity. Suppose that there are constants $K > 0$ and $\beta > 1$, such that*

$$\max_{q=1,\dots,l} h_q(r) \leqslant K r^{\beta} \quad \textit{for all } r \in (0, s_0) \tag{5.6}$$

and for every $q = 1, \dots, l$ exists a $c_q \in \mathbb{R}$, such that

$$\int_0^{s_0} r|\log r| \left| \frac{h_q'(r)}{h_q(r)} - \frac{c_q}{r} \right|^2 dr < \infty. \tag{5.7}$$

We define the set

$$\mathcal{F} := \left\{ \alpha = (\alpha_1, \dots, \alpha_l) \middle| \; \alpha_q \in \{0, 1, \dots, n_q\}, q = 1, \dots, l \right\}.$$

the function $\Xi : \mathcal{F} \to \mathbb{R}$

$$\Xi(\alpha) = \sum_{q=1}^{l} c_q \left(\alpha_q - \frac{n_q}{2} \right) \quad \textit{and} \quad b_\alpha := b_{\alpha_1}(N_1) \cdots b_{\alpha_l}(N_l),$$

where $b_{\alpha_q}(N_q)$ is the α_q-th Betti number of N_q. Under these assumptions

$$\operatorname{ind} D^{GB}_{\min} = \int_{M_1} e + \frac{1}{2} \left(\sum_{\substack{\alpha \in \mathcal{F} \\ (-1)^{|\alpha|}\Xi(\alpha) < 0}} b_\alpha - \sum_{\substack{\alpha \in \mathcal{F} \\ (-1)^{|\alpha|}\Xi(\alpha) \geqslant 0}} b_\alpha \right) - \sum_{\substack{\alpha \in \mathcal{F} \\ -\frac{1}{2} < (-1)^{|\alpha|}\Xi(\alpha) < 0}} b_\alpha,$$

where e denotes the Euler class.

If in addition $\dim M = 4k$, *then*

$$\operatorname{ind} D^{s}_{\min} = \int_{M_1} L_k + \frac{1}{2} \left(\eta\left(\tau' D_N\right) - \sum_{\substack{\alpha \in \mathcal{F} \\ \Xi(\alpha) = 0}} b_\alpha \right) - \sum_{\substack{\alpha \in \mathcal{F} \\ -\frac{1}{2} < \Xi(\alpha) < 0}} b_\alpha,$$

where L_k is the k-th Hirzebruch L-polynomial, $\tau' \alpha_j = (-1)^{k + \frac{(j+1)j}{2}} *_N \alpha_j$ for $\alpha_j \in \Omega^j(N)$ and $D_N = d + \delta$ on $\Omega(N, g_1 \oplus \cdots \oplus g_l)$.

The closed extensions of $D^{GB/s}_{\min}$ are defined by

$$W \subset \mathcal{D}\left(D^{GB/s}_{\max}\right) \Big/ \mathcal{D}\left(D^{GB/s}_{\min}\right) \cong \bigoplus_{\substack{\alpha \in \mathcal{F} \\ |\Xi(\alpha)| < \frac{1}{2}}} \mathcal{H}^{\alpha_1}(N_1) \wedge \ldots \wedge \mathcal{H}^{\alpha_l}(N_l)$$

$$\mathcal{D}\left(D^{GB/s}_{W}\right) := \mathcal{D}\left(D^{GB/s}_{\min}\right) \oplus W, \quad D^{GB/s}_{W} := D^{GB/s}_{\max}\Big|_{\mathcal{D}\left(D^{GB/s}_{W}\right)},$$

where $\mathcal{H}^p(N_q)$ is the vector space of the harmonic p-forms on N_q. The extensions are all Fredholm, and their indices are given by $\operatorname{ind} D^{GB/s}_{W} = \operatorname{ind} D^{GB/s}_{\min} + \dim W$.

Proof. We prove that $D^{GB/s}|_U$ is a horn operator as in Definition 2.1.1. Combining the unitary maps defined in Lemma 5.3.1 with the ones given in Lemma 5.3.3 and 5.3.4, respectively, shows

$$D^{GB/s}|_U \cong \frac{\partial}{\partial r} - T^{GB/s}(r) + A^{GB/s}(r) \text{ on } C_0^\infty((0,s_0],\Omega(N,g_1\oplus\cdots\oplus g_l)).$$

We decompose

$$\begin{aligned}
A^{GB}(r)dx_I &= (-1)^{|I|}\sum_{q=1}^{l}\frac{h_q'}{h_q}\left(|I\cap I_q| - \frac{n_q}{2}\right)dx_I\\
&= \frac{1}{r}(-1)^{|I|}\sum_{q=1}^{l}c_q\left(|I\cap I_q| - \frac{n_q}{2}\right)dx_I + (-1)^{|I|}\sum_{q=1}^{l}\left[\frac{h_q'(r)}{h_q(r)} - \frac{c_q}{r}\right]\left(|I\cap I_q| - \frac{n_q}{2}\right)dx_I\\
&=: \left(\frac{1}{r}A_C^{GB} + A_R^{GB}(r)\right)dx_I,\\
A^{s}(r)dx_I &= \sum_{q=1}^{l}\frac{h_q'}{h_q}\left(|I\cap I_q| - \frac{n_q}{2}\right)dx_I\\
&= \frac{1}{r}\sum_{q=1}^{l}c_q\left(|I\cap I_q| - \frac{n_q}{2}\right)dx_I + \sum_{q=1}^{l}\left[\frac{h_q'(r)}{h_q(r)} - \frac{c_q}{r}\right]\left(|I\cap I_q| - \frac{n_q}{2}\right)dx_I\\
&=: \left(\frac{1}{r}A_C^{s} + A_R^{s}(r)\right)dx_I.
\end{aligned}$$

By Lemma 5.3.3 and Lemma 5.3.4 we know that $\ker T^{GB/s} = \mathcal{H}(N)$ is independent of r. In comparison with Definition 2.1.1, we find

$$\begin{aligned}
&H = \mathcal{H}(N)^\perp, && S(r) = -\, r^\beta T^{GB/s}(r)\Big|_{\mathcal{H}(N)^\perp}, && S_1(r) = A^{GB/s}(r)\Big|_{\mathcal{H}(N)^\perp}\\
&\widetilde{H} = \mathcal{H}(N), && \widetilde{S} = A_C^{GB/s}\big|_{\mathcal{H}(N)}, && \widetilde{S}_1(r) = A_R^{GB/s}(r)\big|_{\mathcal{H}(N)}.
\end{aligned}$$

It remains to be shown that these operators satisfy the asserted properties.

1. *Regularity:* Lemma 5.3.3 and Lemma 5.3.4 imply that the operators

   $$S(r) \in \mathcal{L}(H^1(N,\Lambda(N)), L^2(N,\Lambda(N)))$$

   are self-adjoint. Furthermore, the family $r \mapsto S(r)$ is strongly continuously differentiable since the warping functions are C^1.

2. *Spectral gap:* If $\omega \in (\ker\Delta)^\perp$,

   $$\begin{aligned}
   \|S(r)\omega\|^2 = (S(r)^2\omega,\omega) &= \sum_k |\alpha_k|^2\left(\sum_{q=1}^{l}\left[\frac{r^\beta}{h_q(r)}\right]^2\lambda_q^k\right)\left\|\omega_1^k\wedge\ldots\wedge\omega_l^k\right\|^2\\
   &\overset{(5.6)}{\geqslant} K^{-2}\min\{\lambda_q \mid \lambda_q \in \operatorname{spec}\Delta_q\setminus\{0\},\ q=1,\ldots,l\}\,\|\omega\|^2 =: C_2^2\,\|\omega\|^2.
   \end{aligned}$$

   This implies $|S(r)| \geqslant C_2$.

3. *Spectral projections:* The family $r \mapsto S'(r)S(s_0)^{-1}$ is continuous in norm since the warping functions h_q are C^1. Therefore, the asserted properties follow with Lemma 2.2.2.

4. *Variation:* We do the following computation for the Gauss-Bonnet operator, but the computations for the Signature operator are almost the same.

$$\begin{aligned}\alpha(r) &= \left\|S'(r)S(r)^{-1}\right\| \\ &= \left\| -\sum_{q=1}^{l}\left(\frac{\beta}{r} - \frac{h_q'(r)}{h_q(r)}\right)\frac{r^\beta}{h_q(r)}\widetilde{D}_q\Big|_{\mathcal{H}(N)^\perp}\left(-\sum_{q=1}^{l}\frac{r^\beta}{h_q(r)}\widetilde{D}_q\Big|_{\mathcal{H}(N)^\perp}\right)^{-1}\right\| \\ &\leqslant \sum_{q=1}^{l}\left[\frac{\beta}{r} + \max_{q=1,\dots,l}\left|\frac{h_q'(r)}{h_q(r)}\right|\right]\end{aligned}$$

Thus, $\beta > 1$ and assumption (5.7) imply

$$\int_0^{s_0} r^\beta \alpha^2(r)dr \leqslant 2l\sum_{q=1}^{l}\int_0^{s_0} r^\beta\left[\frac{\beta^2}{r^2} + \left|\frac{h_q'(r)}{h_q(r)}\right|^2\right]dr \overset{\beta>1}{<} \infty.$$

5. *Perturbation S_1:* Formula (5.7) yields

$$\int_0^{s_0} r^\beta \left\|S_1(r)\right\|^2_{\mathcal{L}(H)} dr \leqslant 2l\sum_{q=1}^{l}\int_0^{s_0} r^\beta\left[\left|\frac{h_q'(r)}{h_q(r)} - \frac{c_q}{r}\right|^2 + \frac{c_q^2}{r^2}\right]dr\left(\frac{n_q}{2}\right)^2 \overset{\beta-2>-1}{<} \infty.$$

6. *Cone part $\widetilde{S}$:* $\dim H = \dim \mathcal{H}(N) < \infty$ since N is closed. $\widetilde{S}$ is by definition diagonal and thus symmetric.

7. *Perturbation $\widetilde{S}_1$:* Formula (5.7) yields

$$\int_0^{s_0} r|\log r|\left\|\widetilde{S}_1(r)\right\|^2_{\mathcal{L}(H)} dr \leqslant l\sum_{q=1}^{l}\int_0^{s_0} r|\log r|\left|\frac{h_q'(r)}{h_q(r)} - \frac{c_q}{r}\right|^2 dr\left(\frac{n_q}{2}\right)^2 < \infty.$$

In summary, D^{GB} and D^s are operators with C^1-horn singularity and thus, Theorem 2.1.3 can be applied.

Index formula for the Gauss-Bonnet operator: The Gauss-Bonnet involution τ_N on $\Omega(N)$ is an isomorphism that anti-commutes with D_N and thus gives an isomorphism

$$\tau_N : \ker(D_N - \lambda) \xrightarrow{\sim} \ker(D_N + \lambda) \quad \text{for } \lambda \in \operatorname{spec} D_N \setminus \{0\}.$$

Thus,

$$\eta(S(s_0)) = \eta(-s_0^\beta D_N) = -\eta(D_N) = 0.$$

If $\omega = \omega_1^{\alpha_1} \wedge \ldots \wedge \omega_l^{\alpha_l}$ with $\omega_q^{\alpha_q} \in \mathcal{H}^{\alpha_q}(N_q),\ q = 1, \ldots, l$, then

$$\widetilde{S}\omega = A_C^{GB}\omega = (-1)^{|\alpha|} \sum_{q=1}^{l} c_q \left(\alpha_q - \frac{n_q}{2}\right) \omega_1^{\alpha_1} \wedge \ldots \wedge \omega_l^{\alpha_l} = (-1)^{|\alpha|}\Xi(\alpha)\omega.$$

This implies

$$\sum_{\lambda \geqslant 0} \dim\ker\left(\widetilde{S} - \lambda\right) - \sum_{\lambda < 0} \dim\ker\left(\widetilde{S} - \lambda\right) = \sum_{\substack{\alpha \in \mathcal{F} \\ (-1)^{|\alpha|}\Xi(\alpha) \geqslant 0}} b_\alpha - \sum_{\substack{\alpha \in \mathcal{F} \\ (-1)^{|\alpha|}\Xi(\alpha) < 0}} b_\alpha$$

$$\sum_{-\frac{1}{2} < \lambda < 0} \dim\ker\left(\widetilde{S} - \lambda\right) = \sum_{\substack{\alpha \in \mathcal{F} \\ -\frac{1}{2} < (-1)^{|\alpha|}\Xi(\alpha) < 0}} b_\alpha.$$

Index formula for the Signature operator:

$$\eta(S(s_0)) = \eta(-s_0^\beta \tau' D_N) = -\eta(\tau' D_N)$$

If $\omega = \omega_1^{\alpha_1} \wedge \ldots \wedge \omega_l^{\alpha_l}$ with $\omega_q^{\alpha_q} \in \mathcal{H}^{\alpha_q}(N_q),\ q = 1, \ldots, l$, then

$$\widetilde{S}\omega = A_C^{s}\omega = \sum_{q=1}^{l} c_q \left(\alpha_q - \frac{n_q}{2}\right) \omega_1^{\alpha_1} \wedge \ldots \wedge \omega_l^{\alpha_l} = \Xi(\alpha)\omega.$$

Since

$$b_{(n_1-\alpha_1,\ldots,n_l-\alpha_l)} = \prod_{q=1}^{l} b_{n_q-\alpha_q}(N_q) = \prod_{q=1}^{l} b_{\alpha_q}(N_q) = b_{(\alpha_1,\ldots,\alpha_l)}$$

and

$$\Xi\big(\,(n_1-\alpha_1,\ldots,n_l-\alpha_l)\,\big) = \sum_{q=1}^{l} c_q\left(n_q - \alpha_q - \frac{n_q}{2}\right) = -\sum_{q=1}^{l} c_q\left(\alpha_q - \frac{n_q}{2}\right) = -\Xi(\alpha),$$

it follows

$$\sum_{\lambda \geqslant 0} \dim\ker\left(\widetilde{S} - \lambda\right) - \sum_{\lambda < 0} \dim\ker\left(\widetilde{S} - \lambda\right) = \sum_{\substack{\alpha \in \mathcal{F} \\ \Xi(\alpha) \geqslant 0}} b_\alpha - \sum_{\substack{\alpha \in \mathcal{F} \\ \Xi(\alpha) < 0}} b_\alpha = \sum_{\substack{\alpha \in \mathcal{F} \\ \Xi(\alpha) = 0}} b_\alpha$$

$$\sum_{-\frac{1}{2} < \lambda < 0} \dim\ker\left(\widetilde{S} - \lambda\right) = \sum_{\substack{\alpha \in \mathcal{F} \\ -\frac{1}{2} < \Xi(\alpha) < 0}} b_\alpha.$$

In both cases Section 4 in [APS75] page 60f. which refers to the article [ABP73] (note also [ABP75]), implies $\operatorname{Res}_k \eta_{\widetilde{S}} = 0$ for all $k \geqslant 1$. $\square$

Examples:

1. *Manifolds with multiple metric horns:* Let $h_q \in C^1((0,s_0],(0,\infty))$ with $h_q(r) = r^{\beta_q}$, $\beta_q > 1$ for r close to 0 for all $q \in \{1,\dots,l\}$. The warping functions satisfy assertion (5.6):

$$\max_{q=1,\dots,l} h_q(r) \leqslant K r^{\beta}, \qquad \text{where } \beta = \min_{q\in\{1,\dots,l\}} \beta_q > 1.$$

For $c_q := \beta_q$, the functions also satisfy assertion (5.7):

$$\int_0^{s_0} r|\log r| \left|\frac{h_q'(r)}{h_q(r)} - \frac{\beta_q}{r}\right|^2 dr = C + \int_0^{s_1} r|\log r| \left|\frac{\beta_q}{r} - \frac{\beta_q}{r}\right|^2 dr = C < \infty.$$

Thus, Theorem 5.3.5 applies with

$$\Xi(\alpha) = \sum_{q=1}^{l} \beta_q \left(\alpha_q - \frac{n_q}{2}\right).$$

Note that no other assumptions than $\beta_q > 1$ for $q = 1,\dots,l$ have to be made.

2. *Manifolds with metric horns:* This is a special case of the first example: $l = 1$ and $h_1(r) = r^{\beta}$ close to zero. Then

$$\Xi : \{0,\dots,n\} \ni \alpha \mapsto \beta\left(\alpha - \frac{n}{2}\right).$$

If n is odd, i.e. $\dim M = n+1$ is even, then $|\Xi(\alpha)| \geqslant \frac{\beta}{2} > \frac{1}{2}$ and thus $D^{GB/s}_{\min} = D^{GB/s}_{\max}$. The index of the Gauss-Bonnet operator for manifolds with metric horns has been computed in Theorem 6.1 and Theorem 6.5 in [LP98], and the index of the Signature operator on manifolds with metric horns is given as Theorem 1.3 in [Brü96].

Bibliography

[ABP73] M. F. Atiyah, R. Bott, and V. K. Patodi, *On the heat equation and the index theorem*, Invent. Math. **19** (1973), 279–330.

[ABP75] M. Atiyah, R. Bott, and V. K. Patodi, *Errata to: "On the heat equation and the index theorem" (Invent. Math.* **19** *(1973), 279–330)*, Invent. Math. **28** (1975), 277–280.

[APS75] M. F. Atiyah, V. K. Patodi, and I. M. Singer, *Spectral asymmetry and Riemannian geometry. I*, Math. Proc. Cambridge Philos. Soc. **77** (1975), 43–69.

[BB01] W. Ballmann and J. Brüning, *On the spectral theory of manifolds with cusps*, J. Math. Pures Appl. (9) **80** (2001), no. 6, 593–625.

[BL96] J. Brüning and M. Lesch, *On the spectral geometry of algebraic curves*, J. Reine Angew. Math. **474** (1996), 25–66.

[Brü88] J. Brüning, *Heat kernel asymptotics for operator valued Sturm-Liouville equations*, Analysis **8** (1988), no. 1-2, 73–93.

[Brü90] ———, *L^2-index theorems on certain complete manifolds*, J. Differential Geom. **32** (1990), no. 2, 491–532.

[Brü92] ———, *On L^2-index theorems for complete manifolds of rank-one type*, Duke Math. J. **66** (1992), no. 2, 257–309.

[Brü96] ———, *The signature theorem for manifolds with metric horns*, Journées "Équations aux Dérivées Partielles" (Saint-Jean-de-Monts, 1996), École Polytech., Palaiseau, 1996, pp. Exp. No. II, 10.

[BS87] J. Brüning and R. Seeley, *The Resolvent Expansion for Second Order Regular Singular Operators*, J. Funct. Anal. **73** (1987), 369–429.

[BS88] ———, *An index theorem for first order regular singular operators*, American Journal of Mathematics **110** (1988), 659–714.

[Che80] J. Cheeger, *On the Hodge theory of Riemannian pseudomanifolds*, Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979), Proc. Sympos. Pure Math., XXXVI, Amer. Math. Soc., Providence, R.I., 1980, pp. 91–146.

[Cho85] A. W. Chou, *The Dirac operator on spaces with conical singularities and positive scalar curvatures*, Transactions of the American Mathematical Society **289** (1985), 1–40.

[CL63] H. O. Cordes and J. P. Labrousse, *The invariance of the index in the metric space of closed operators*, J. Math. Mech. **12** (1963), 693–719.

[dC92] M. P. do Carmo, *Riemannian Geometry*, Birkhäuser, 1992.

[Gil95] P. B. Gilkey, *Invariance Theory, the Heat Equation and the Atiyah-Singer Index Theorem, 2nd Edition*, CRC Press, Boca Raton, 1995.

[Hö85] L. Hörmander, *The Analysis of Linear Partial Differential Operators III*, Springer-Verlag Berlin Heidelberg, 1985.

[Kat95] T. Kato, *Perturbation theory for linear operators*, Classics in Mathematics, Springer-Verlag, Berlin, 1995, Reprint of the 1980 edition.

[Kis64] J. Kisyński, *Sur les opérateurs de Green des problèmes de Cauchy abstraits*, Studia Math. **23** (1963/1964), 285–328.

[Kli03] F. Klinker, *The spinor bundle of Riemannian products*, arXiv:math/0212058.

[Kre71] S. G. Kreĭn, *Linear differential equations in Banach space*, American Mathematical Society, Providence, R.I., 1971, Translated from the Russian by J. M. Danskin, Translations of Mathematical Monographs, Vol. 29.

[LM89] H. B. Lawson and M.-L. Michelson, *Spin Geometry*, Princeton University Press, 1989.

[LP98] M. Lesch and N. Peyerimhoff, *On index formulas for manifolds with metric horns*, Comm. Partial Differential Equations **23** (1998), no. 3-4, 649–684.